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Transport Coefficients for Magnetohydrodynamic Cellular Automata

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Abstract. A Chapman-Enskog development has been used to infer theoretical expressions for coefficients of kinematic viscosity and magnetic diffusivity for a two-dimensional magnetohydrodynamic cellular automaton.

1. Introduction

This article will present some approximate theoretical calculations of transport coefficients for magnetohydrodynamic (MHD) cellular automata (CA). Throughout, the attention will be limited to two-dimensional (2D) models.

CA, as possible fluid simulation tools, have recently become an active research area [1–4], following several years of basic background investigations on lattice gases [5–8]. A cellular automaton for simulating 2D MHD has recently been proposed [9]. As in the fluid case, 2D MHD macroscopic continuum equations can be derived with some degree of conviction from the dynamics represented by the microscopic computer game which the cellular automaton is. In interpreting future results from MHD CA computations, it will be desirable to have theoretical estimates of the kinematic viscosity and magnetic diffusivity derived from a microscopic kinetic theory. The principal purpose of this article is to present such a derivation.

Section 2 reviews the 2D MHD CA previously put forward [9], and relates it to its ancestor, the hexagonal lattice gas [1]. Section 3 presents a Markovian stochastic model of its dynamics. Section 4 is devoted to the calculation of the kinematic viscosity ν and the magnetic diffusivity η, which are expected to characterize the magnetofluid which the CA is simulating. Section 5 briefly discusses the results and remarks upon possible generalizations. In the development, some familiarity with the Chapman-Enskog procedure for deriving fluid equations and transport coefficients from kinetic equations will be assumed; the books of Chapman and Cowling

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[10], or Ferziger and Kaper [11] are valuable expositions of the Chapman-Enskog method. The systematic derivation of kinetic equations from a more basic description such as Liouville's equation [12,13] is not as far advanced as it is in classical continuum kinetic theory. The kinetic description used is of a more provisional character [2,4].

2. 2D MHD CA

Many properties of the 2D MHD CA [9] are appropriated from the hexagonal lattice gas CA of Frisch, Hasslacher, and Pomeau [1]. Our presentation draws heavily on a treatment of that model due to Wolfram [2]. Both models have particles which reside at the centers of adjacent hexagons in one of six discrete velocity states, $\hat{e}_a = (\cos(2\pi a/6),\ \sin(2\pi a/6))$, $a = 1, 2, \ldots 6$. The particles all move to the centers of the adjacent hexagons toward which they are directed, at discretized integer time steps. Between time steps, a set of "scattering rules" permit the particles to make transitions among the different $\hat{e}_a$ within a given hexagon. These scattering rules conserve momentum, and (trivially) conserve particle number and kinetic energy. Fermi-Dirac statistics apply, so that no more than one particle per single-particle state per hexagon is ever permitted. Macroscopic fluid variables such as density or fluid velocity are interpreted as averages of these quantities over the particles inside "supercells" consisting of many adjacent hexagons. Existing pilot codes typically have ten million or more hexagons and a thousand or more supercells.

All the properties in the previous paragraph are shared by the 2D MHD and two-dimensional Navier-Stokes models, although the 2D MHD model [9] adds two features to the two-dimensional Navier-Stokes model. First, the particles carry an additional integer index σ corresponding to the usual [14] 2D MHD one-component magnetic vector potential $\mathbf{A} = A_z(x,\ y,\ t)\hat{e}_z$. σ is allowed to take on the values ± 1 or 0; thus, there become 18 allowed particle states per hexagon instead of 6. The geometry is such that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = (B_x,\ B_y,\ 0)$, the fluid velocity $\mathbf{u} = (\mathbf{u}_x,\ \mathbf{u}_y\ 0)$, and all variables are independent of z. The σ label has variously been alluded to as a "photon" index, a "spin", or a "color"; we will use the term photon index as corresponding most closely to the physics being described. Second, a Lorentz force $(\nabla \times \mathbf{B})\ \times \mathbf{B} = -(\nabla A_z)\nabla^2 A_z$ is introduced into the equation of motion in the same way that external forces such as gravity are presently introduced in the hydrodynamic case. This involves stepping outside the pure CA framework to impart the requisite momentum per unit volume by randomly flipping the microscopic distribution over the $\hat{e}_a$ proportionally to $-(\nabla A_z)\nabla^2 A_z$. (For purposes of computing this Lorentz force, standard finite-difference approximations are applied to the supercell averages of σ, which are interpreted as A_z.)

The just-described model has been shown [9] to lead to the following standard set of two-dimensional magnetofluid equations for low Mach numbers ($u^2 \ll 1$):

$$n\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla\cdot\mathbf{p}+\mathbf{j}\times\mathbf{B}+n\nu\nabla^2\mathbf{u}, \tag{2.1}$$

$$\frac{\partial A_z}{\partial t}+\mathbf{u}\cdot\nabla A_z=\eta\nabla^2 A_z, \tag{2.2}$$

$$\nabla\cdot\mathbf{u}=-\frac{1}{n}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)n\cong 0. \tag{2.3}$$

The symbols in equations (2.1) through (2.3) mean the following. The field variables are understood as supercell averages over several (typically, $(32)^2$) hexagons adjacent to each other. n is the particle number density. n has a maximum of 18 per hexagon ($\cong 20.8$ per unit area) and has been assumed in equations (2.1) through (2.3) to be $\ll 9$ per hexagon. $\mathbf{u}$ is the fluid velocity and has been assumed to be small in magnitude compared to unity. $\mathbf{j}=j_z\hat{\mathbf{e}}_z=\nabla\times\mathbf{B}=-(\nabla^2 A_z)\hat{\mathbf{e}}_z$ is the (supercell-averaged) electric current density. $\mathbf{j}\times\mathbf{B}=-(\nabla A_z)\nabla^2 A_z$ is the Lorentz volume force. The pressure dyadic is $\mathbf{p}\cong(n/2)\mathbf{1}(1-u^2/2)-n\mathbf{u}\mathbf{u}/2$. (Here, as elsewhere, more complicated coefficients result unless n is $\ll 9$.)

Finally, ν and η are the coefficients of kinematic viscosity and magnetic diffusivity respectively. This paper is devoted to an approximate theory of their evaluation for the case in which the effect of the Lorentz force on the distribution function is small compared to the effects of particle collisions. Once a kinetic equation is written down, such as a Boltzmann equation, Chapman-Enskog methods can be used to extract transport coefficients [2,10]. This program is carried out in sections 3 and 4.

3. Markovian stochastic model

The exact one-particle distribution function is discrete. For each hexagon, it consists of 18 numbers, each either 0 or 1, depending upon whether there is or is not present a particle with velocity $\hat{\mathbf{e}}_a$ and photon label σ. For fluid calculations, we require a smooth, differentiable distribution function, and we get it by ensemble averaging the exact one. The realizations of the ensemble remain discrete in velocity $\hat{\mathbf{e}}_a$ and photon label σ, and differ from each other only by spatial translations, and by temporal translations of the instant which is called $t=0$. The ensemble-averaged distribution function, $f_{a,\sigma}(\mathbf{x},t)$, may be thought of as an 18-component vector field ($a=1,2,\ldots 6$, and $\sigma=-1,\ 0,\ +1$) which is a continuous function of space coordinates $\mathbf{x}$ and time t, now themselves both continuous. Always, $0\le f_{a,\sigma}\le 1$. In terms of $f_{a,\sigma}$, the various fluid variables are defined as moments, according to the definitions

$$lln=\sum_{a,\sigma} f_{a,\sigma}\qquad\text{(number density)} \tag{3.1}$$

$$nu = \sum_{a,\sigma} \hat{e}_a f_{a,\sigma} \qquad \text{(fluid velocity} = u) \tag{3.2}$$

$$nA_z = \sum_{a,\sigma} \sigma f_{a,\sigma} \qquad \text{vector potential} = A_z) \tag{3.3}$$

$$p = \sum_{a,\sigma} (\hat{e}_a - u)(\hat{e}_a - u) f_{a,\sigma} \qquad \text{(pressure tensor)} \tag{3.4}$$

$$\pi = \sum_{a,\sigma} \hat{e}_a \hat{e}_a f_{a,\sigma} \quad \text{(momentum flux tensor).} \tag{3.5}$$

$$\phi = \sum_{a,\sigma} \hat{e}_a \sigma f_{a,\sigma} \qquad A_z\text{—flux vector).} \tag{3.6}$$

In equations (3.1) through (3.6), the $\sum_a$ always runs over $a = 1$ to 6, and the $\sum_\sigma$ always runs over $-1, 0, +1$.

Though it is not a deduction, it is a reasonable assumption that $f_{a,\sigma}$ may be advanced from its value at a previous time step according to the Markovian recipe:

$$f_{a,\sigma}(\mathrm{x},\ t) =$$

$$\sum_{b,\lambda} \int d\mathrm{x}' P(a,\ \sigma,\ \mathrm{x},\ t | b,\ \lambda,\ \mathrm{x}',\ t-1) f_{b,\lambda}(\mathrm{x}',\ t-1). \tag{3.7}$$

P is an assumed probability for a particle with coordinates $\hat{e}_b$, λ, x' at time $t-1$ to find itself with coordinates $\hat{e}_a$, σ, x at time t. Until an explicit expression for P is given, of course, the content of equation (3.7) is only formal.

Spatial translations from one hexagon to another are always regarded as sharp, so that $\mathrm{x}' = \mathrm{x} - \hat{e}_b$. In the instant before time t, collisions are understood to occur which may scatter the particle from $(\hat{e}_b,\ \lambda)$ to $(\hat{e}_a,\ \sigma)$. All possibilities $\hat{e}_b, \lambda$ are understood to be summed over. In the spirit of Boltzmann, the transition probability P may be considered to depend upon the distribution function f itself, which will make the right-hand side of equation (3.7) nonlinear in f. (We will frequently omit subscripts and arguments when they are obvious from the context.)

Under the above assumptions, equation (3.7) reduces to

$$f_{a,\sigma}(\mathrm{x},\ t) = \sum_{b,\lambda} P_{a\sigma;b\lambda}(\mathrm{x},\ t) f_{b,\lambda}(\mathrm{x} - \hat{e}_b,\ t-1). \tag{3.8}$$

In the absence of collisions, $P_{a\sigma;b\lambda} = \delta_{ab}\delta_{\sigma\lambda}$. Subtracting the collisionless version of equation (3.8) from both sides gives

$$f_{a,\sigma}(\mathrm{x},\ t) - f_{a,\sigma}(\mathrm{x} - \hat{e}_a,\ t-1) =$$

$$\sum_{b,\lambda} \left(P_{a\sigma;b\lambda}(\mathrm{x},\ t) - \delta_{ab}\delta_{\sigma\lambda}\right) f_{b\lambda}(\mathrm{x} - \hat{e}_b,\ t-1). \tag{3.9}$$

For particle-conserving transition probabilities, which is what we are dealing with here, it is useful to rewrite P as

$$P_{a\sigma;b\lambda}(\mathbf{x},\ t) = \delta_{ab}\delta_{\sigma\lambda}\left[1 - \sum_{c,\mu} W_{c\mu;b\lambda}(\mathbf{x},\ t)\right] + W_{a\sigma;b\lambda}(\mathbf{x},\ t) \tag{3.10}$$

in terms of a non-negative transition probability W.

We insert equation (3.10) into equation (3.9), and further make the assumption that $f_{a,\sigma}(\mathbf{x},\ t)$ varies over characteristic lengths and times large compared to unity. This latter assumption permits us to Taylor expand the second term on the left-hand side of equation (3.9) with respect to $\mathbf{x}$ and t. The zeroth-order terms cancel, and we are left with

$$\begin{aligned}&\left[\left(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla\right) - \frac{1}{2}\left(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla\right)^2 + \ldots\right] f_{a,\sigma}(\mathbf{x},\ t) \\ &= \sum_{b,\lambda} [-W_{b\lambda;a\sigma}(\mathbf{x},\ t) f_{a,\sigma}(\mathbf{x} - \hat{e}_a,\ t-1) \\ &\quad + W_{a\sigma;b\lambda}(\mathbf{x},\ t) f_{b,\lambda}(\mathbf{x} - \hat{e}_b,\ t-1)]\,. \end{aligned} \tag{3.11}$$

In terms of the conventional usage applied to the Boltzmann equation, the terms on the right-hand side of equation (3.11) correspond to the effects of "direct" and "inverse" collisions, respectively.

If we further Taylor-expand the f's on the right-hand side of equation (3.11) and equate the leading terms on both sides to each other, we get

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla\right) f_{a,\sigma}(\mathbf{x},\ t) = \sum_{b,\lambda} [-W_{b\lambda;a\sigma}(\mathbf{x},\ t) f_{a,\sigma}(\mathbf{x},\ t) \\ &\quad + W_{a\sigma;b\lambda}(\mathbf{x},\ t) f_{b,\lambda}(\mathbf{x},\ t)] \equiv \Omega_{a\sigma}(f). \end{aligned} \tag{3.12}$$

Equation (3.12) is the starting point for our Chapman-Enskog development. The right-hand side determines $\Omega_{a\sigma}(f)$, which we call the collision integral. Its content is still formal until we adopt explicit expressions for the W's. Also absent from our present considerations is the effect of the higher-order terms in the Taylor expansions which are dropped in getting from equations (3.11) to (3.12); it may be that modifications to the transport coefficients will result from them [2,15]. This should be regarded as an important modification to be considered in the future.

The W's must always be chosen to satisfy the three non-trivial conservation laws:

$$\sum_{a,\sigma} \Omega_{a\sigma} = 0, \tag{3.13}$$

$$\sum_{a,\sigma} \hat{e}_a \Omega_{a\sigma} = 0, \tag{3.14}$$

$$\sum_{a,\sigma} \sigma \Omega_{a\sigma} = 0, \tag{3.15}$$

for all $f_{a,\sigma}$, in addition to the trivial conservation law for kinetic energy. We defer until section 4 writing down explicit forms for equation (3.12) which will satisfy equations (3.13) through (3.15).

It will be assumed that the net effect of $\Omega_{a\sigma}$ in equation (3.12) is to drive $f_{a,\sigma}$ to a local thermal equilibrium over length and time scales short compared to those over which n, $\mathbf{u}$, $\mathbf{p}$, A_z, etc., vary. This makes possible an iterative Chapman-Enskog approach to equation (3.12):

$$f_{a,\sigma}(\mathbf{x},\, t) = f_{a,\sigma}^{(0)}(\mathbf{x},\, t) + f_{a,\sigma}^{(1)}(\mathbf{x},\, t) + \ldots, \tag{3.16}$$

where $f_{a,\sigma}^{(0)}(\mathbf{x},\, t)$ is the local Fermi-Dirac distribution, which depends on $\mathbf{x}$ and t only functionally through dependence on $n(\mathbf{x},\, t)$, $\mathbf{u}(\mathbf{x},\, t)$, and $A_z(\mathbf{x},\, t)$. $f_{a,\sigma}^{(1)}$ is the first-order deviation of $f_{a,\sigma}$ from $f_{a,\sigma}^{(0)}$, where the parameter of smallness is the usual one: the ratio of the collision mean-free path to the characteristic length scale for the variation of the moments, or the mean-free time to the time scale for the variation of the moments.

Since $\Omega_{a\sigma}(f^{(0)}) = 0$, the first-order terms in the smallness parameter are $(\partial/\partial t + \hat{\mathbf{e}}_a \cdot \nabla) f_{a,\sigma}^{(0)}$ on the left-hand side and the $f^{(1)}$-proportional part of the right-hand side, which we abbreviate as $\Omega_{a\sigma}^{(1)}$:

$$\left(\frac{\partial}{\partial t} + \hat{\mathbf{e}}_a \cdot \nabla\right) f_{a,\sigma}^{(0)}(\mathbf{x},\, t) = \Omega_{a\sigma}^{(1)} = \sum_{b,\lambda} C_{a\sigma,b\lambda}^{(0)} f_{b,\lambda}^{(1)}(\mathbf{x},\, t). \tag{3.17}$$

The right-hand side of equation (3.17) stands symbolically for the result of linearizing the right-hand side of equation (3.12) about $f_{a,\sigma}^{(0)}$ in powers of $f_{a,\sigma}^{(1)}$. Explicit forms for $\Omega_{a\sigma}^{(1)}$ and $C_{a\sigma,b\lambda}^{(0)}$ will be given in section 4 and the appendices. $C_{a\sigma,b\lambda}^{(0)}$ is in effect an 18×18 collision matrix. Some difficulty is involved in inverting it, but that is what is required in order to express $f_{a,\sigma}^{(1)}$ in terms of $(\partial/\partial t + \hat{\mathbf{e}}_a \cdot \nabla) f_{a,\sigma}^{(0)}$. Once we have $f_{a,\sigma}^{(1)}$, we may express such first-order moments as

$$\pi^{(1)} = \sum_{a,\sigma} \hat{\mathbf{e}}_a \hat{\mathbf{e}}_a f_{a,\sigma}^{(1)}(\mathbf{x},\, t), \tag{3.18}$$

and

$$\phi^{(1)} = \sum_{a,\sigma} \hat{\mathbf{e}}_a \sigma f_{a,\sigma}^{(1)}(x,\, t), \tag{3.19}$$

in terms of the (unexpanded) moments n, $\mathbf{u}$, A_z.

The three exact conservation laws which follow from equations (3.12) and (3.13) through (3.15) are:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \tag{3.20}$$

$$\frac{\partial (n\mathbf{u})}{\partial t} + \nabla \cdot \pi = 0, \tag{3.21}$$

$$\frac{\partial}{\partial t}(nA_z) + \nabla \cdot \phi = 0. \tag{3.22}$$

If the moments in equations (3.20) through (3.22) are evaluated using only $f^{(0)}_{a,\sigma}$, the ideal Euler equations result. The expressions (3.18) and (3.19) add dissipative corrections to (3.21) and (3.22) which, in the usual way, lead to equations (2.1) through (2.3). The reader is reminded that the Lorentz force, $\mathbf{j} \times \mathbf{B}$, in equation (2.1) has to be put in as an "external" force in the way already indicated.

4. Evaluation of Chapman-Enskog transport coefficients

In this section, we first specialize $\Omega_{a,\sigma}$ to a Boltzmann-like collision term which permits two- and three-body collisions. In the notation of Wolfram [2], the scattering events allowed are the $2L$, $2R$ and $3S$ collisions. In addition to these mechanical collisions, simultaneous σ-conserving transitions of the photon index are allowed and may change the numbers of the $+1, 0, or -1$ σ-values in some collisions [9]. Second, we linearize this collision integral about a local thermodynamic equilibrium distribution and calculate the first Chapman-Enskog correction to the distribution function. Finally, we use this calculated distribution function, substituting it into the exact differential form of the conservation laws, to infer coefficients of kinematic viscosity and magnetic diffusivity. The calculation is algebraically lengthy and tedious and the details are relegated to four appendices, to which the reader who is primarily interested in the results may wish to skip.

The shorthand notation $\tilde{f}_{a,\sigma} = f_{a,\sigma}/(1-f_{a,\sigma})$ is adopted, and the collision term is of the generic form [2]:

$$\left(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla\right) f_{a,\sigma} = \{\Pi_{b,\lambda}(1 - f_{b,\lambda})\} \left\{ \sum_{\mu\xi} S^{2L}_{\mu\xi}(\sigma) \tilde{f}_{a+1,\mu}\tilde{f}_{a+4,\xi} \right.$$
$$+ \sum_{\mu\xi} S^{2R}_{\mu\xi}(\sigma)\tilde{f}_{a+2,\mu}\tilde{f}_{a+5,\xi} - \sum_{\mu} S^{2}_{\mu}(\sigma)\tilde{f}_{a,\sigma}\tilde{f}_{a+3,\mu}$$
$$\left. + \sum_{\mu\xi\gamma} S^{3}_{\mu\xi\gamma}(\sigma)\tilde{f}_{a+1,\mu}\tilde{f}_{a+3,\xi}\tilde{f}_{a+5,\gamma} - \sum_{\mu\xi} S^{3}_{\mu\xi}(\sigma)\tilde{f}_{a,\sigma}\tilde{f}_{a+2,\mu}\tilde{f}_{a+4,\xi} \right\}. \tag{4.1}$$

The coefficients $S^{2L}_{\mu\xi}(\sigma)$, $S^{2R}_{\mu\xi}$, $S^{2}_{\mu}(\sigma)$, ... in front of the distribution-function products in equation (4.1) are numbers which play the role of the differential scattering cross sections in the usual continuous Boltzmann equation. They are to some extent arbitrary and depend upon how the CA are set up. They are chosen to vanish for any collision which does not satisfy the conservation laws. Their numerical values are constrained by detailed balance considerations and by symmetry requirements. The terms with positive signs in equation (4.1) add particles to the states a, σ, and the terms with negative signs remove particles from a, σ. H-theorems have been proved for simpler but similar collision integrals [8], and it is reasonable to assume that an H-theorem also holds for equation (4.1). It may be readily verified that the form for $\Omega_{a,\sigma}$ given in detail in Appendix A vanishes when we set $f_{a,\sigma} = f_{a,\sigma}^{(0)} \equiv [1 + \exp(\alpha + \beta \mathbf{u} \cdot \hat{e}_a + \gamma\sigma A_z)]^{-1}$, where the coefficients in $f_{a,\sigma}^{(0)}$ are arbitrary. These Lagrange multipliers α, β, γ are determined locally in terms of n, $\mathbf{u}$ and A_z by equations (3.1) to (3.3), each moment being permitted a slow $\mathbf{x}$ and t dependence. $\Omega_{a\sigma}(f)$ is separated into four terms, $\Omega_{a\sigma}(f) = \Omega_{a\sigma}^{I} + \Omega_{a\sigma}^{II} + \Omega_{a\sigma}^{III} + \Omega_{a\sigma}^{IV}$, given in Appendix A, with all coefficients enumerated.

We seek the second term of the Chapman-Enskog solution by solving (3.17) for $f_{a,\sigma}^{(1)}$. This solution is then fed, in turn, into equations (3.18) through (3.19) to give first-order dissipative contributions to equations (3.21) and (3.22).

The linearized collision term from equation (3.17) is identified in detail in Appendix B. It is found that the collision matrix in equation (3.17) whose element is $C_{a\sigma,b\lambda}^{(0)}$ can be written as the sum of four direct products of matrices which act in the $\hat{e}_a$ space and the σ space separately. To represent this conveniently, we write the matrix as

$$\vec{\vec{\mathbf{C}}}^{(0)} = \sum_{j=1}^{4} \vec{\vec{\omega}}^{(j)} \otimes \boldsymbol{\tau}^{(j)}. \tag{4.2}$$

Arrows over vectors and dyads remind the reader that they refer to the six-dimensional $\hat{e}_a$-space, and boldface vectors and dyads without arrows refer to the three-dimensional σ-space. In component notation, we have that

$$\left(\vec{\vec{\omega}}^{(j)} \otimes \boldsymbol{\tau}^{(j)}\right)_{a\sigma,b\lambda} = \omega_{ab}^{(j)} \tau_{\sigma\lambda}^{(j)}, \tag{4.3}$$

(a and b run from 1 to 6, and σ and λ run from -1 to +1). All the $\vec{\vec{\omega}}^{(j)}$, $j = 1$, 2, 3, 4, are 6×6 circulant matrices [2,16], and represent scattering processes for velocity, while all $\boldsymbol{\tau}^{(j)}$, $j = 1, 2, 3, 4$, represent 3×3 matrices describing scattering events for the photon index σ, but are not all circulant. The linearization of $\Omega_{a\sigma}$ is tedious, and leads to expressions for the $\vec{\vec{\omega}}^{(j)}$ and $\boldsymbol{\tau}^{(j)}$ which are enumerated in Appendix B. They are given there for the limit in which both $\mathbf{u}$ and A_z are small compared to unity.

The inversion of the collision matrix $\vec{\vec{\mathbf{C}}}^{(0)}$ is made possible by the fact that all circulant matrices of a given dimension M (here, $M = 6$) have the same set of right eigenvectors [2,16] (see Appendix C). Use of this fact will reduce the inversion of the (18×18) matrix $\vec{\vec{\mathbf{C}}}^{(0)}$ to that of inverting a new matrix which is only 3×3.

The circulant matrices $\vec{\vec{\omega}}^{(i)}$ have the same set of eigenvectors $\vec{v}^{c}$, and eigenvalues $\lambda^{c(i)}$ (see Appendix C):

$$\vec{\omega}^{(i)} \cdot \vec{v}^c = \lambda^{c(i)} \vec{v}^c, \tag{4.4}$$

where $i = 1, 2, 3, 4$ and c runs from 1 to 6. We may expand the $f_{b,\lambda}^{(1)}$ in equation (3.17) in terms of the $\vec{v}^c$, so that

$$f_{b,\lambda}^{(1)} = \sum_c \psi_\lambda^c v_b^c, \tag{4.5}$$

so that finding the expansion coefficients, ψ_λ^c, is equivalent to finding $f_{b,\lambda}^{(1)}$.

Using equation (4.5) in $\Omega_{a\sigma}$, we have, because of equations (4.2) and (4.3),

$$\sum_{b,\lambda} C_{a\sigma,b\lambda}^{(0)} f_{b,\lambda}^{(1)} = \sum_\lambda \sum_c \tau_{\sigma\lambda}^c \psi_\lambda^c v_a^c, \tag{4.6}$$

in terms of a new matrix τ^c whose elements are defined by

$$\tau_{\sigma\lambda}^c = \sum_{j=1}^{4} \lambda^{c(j)} \tau_{\sigma\lambda}^{(j)}. \tag{4.7}$$

We need to find eigenvectors $w_\tau^{c,\nu}$ ($\nu = 1, 2, 3$) and corresponding eigenvalues $\xi^{c,\nu}$ for τ^c

$$\sum_{\lambda=1}^{3} \tau_{\sigma\lambda}^c w_\lambda^{c,\nu} = \xi^{c,\nu} w_\sigma^{c,\nu}. \tag{4.8}$$

This is done in Appendix D.

The three-component vector ψ^c is represented in terms of the eigenvectors $\mathbf{w}^{c,\nu}$ ($\nu = 1, 2, 3$) as

$$\psi_\lambda^c = \sum_{\nu=1}^{3} \rho^{c,\nu} w_\lambda^{c,\nu}, \tag{4.9}$$

so that

$$\sum_{b,\lambda} C_{a\sigma,b\lambda}^{(0)} f_{b,\lambda}^{(1)} = \sum_{c=1}^{6} \sum_{\nu=1}^{3} v_a^c w_\sigma^{c,\nu} \xi^{c,\nu} \rho^{c,\nu}. \tag{4.10}$$

The solution of equation (3.17) is then

$$f_{a,\sigma}^{(1)} = \sum_{c=1}^{6} \sum_{\nu=1}^{3} v_a^c w_\sigma^{c,\nu} \rho^{c,\nu}, \tag{4.11}$$

with the coefficient $\rho^{c,\nu}$ given by

$$\rho^{c,\nu} = \frac{1}{\xi^{c,\nu}} \sum_{a=1}^{6} \sum_{\sigma=-1}^{+1} (v_a^c w_\sigma^{c,\nu})^* \left(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla \right) f_{a,\sigma}^{(0)}. \tag{4.12}$$

4.1 Evaluation of the viscosity

Using the Euler equations [9] obtained from putting $f^{(0)}_{a,\sigma}$ into equations (3.20) through (3.22), we have, for the left-hand side of equation (3.17),

$$\left(\frac{\partial}{\partial t} + \hat{e}_a \cdot \nabla\right) f^{(0)}_{a,\sigma} = \frac{(2\hat{e}_a\hat{e}_a - \mathbf{1})}{18} : n\nabla\mathbf{u} + \frac{\sigma\hat{e}_a}{12} n \cdot \nabla A_z. \tag{4.13}$$

It is the first term on the right-hand side of (4.13) which makes a non-vanishing contribution to the viscosity; it leads to a contribution to $\rho^{c,\nu}$ of

$$\rho^{c,\nu}(\text{visc.}) = \frac{1}{\xi^{c,\nu}} \sum_{a,\sigma} (v^c_a w^{c,\nu}_\sigma)^* \left(\frac{2\hat{e}_a\hat{e}_a - \mathbf{1}}{18}\right) : \ n\nabla\mathbf{u}. \tag{4.14}$$

It can be shown using the material in Appendix C that

$$\sum_{a=1}^{6} (v^c_a)^* (2\hat{e}_a\hat{e}_a - \mathbf{1}) = 0, \qquad \text{for } c = 1,2,4,6, \tag{4.15}$$

so that coefficients $\rho^{c,\nu}(\text{visc.})$ survive only for $c = 3$ and 5. Moreover, the eigenvectors $w^{c,\nu}$ (Appendix D) have symmetries such that

$$\sum_\sigma w^{3,\nu}_\sigma = \sum_\sigma w^{5,\nu}_\sigma = 0 \qquad \text{for } \nu = 1,2. \tag{4.16}$$

It follows that only $\rho^{3,3}(\text{visc.})$ and $\rho^{5,3}(\text{visc.})$ remain finite.

Inserting the solution (4.14) into equation (3.18) and using the results just quoted yields

$$\begin{aligned}\pi^{(1)} &= \sum_{a=1}^{6} \sum_{\sigma=1}^{3} \hat{e}_a\hat{e}_a \left(v^3_a w^{3,3}_\sigma \rho^{3,3} + v^5_a w^{5,3}_\sigma \rho^{5,3}\right) \\ &= \left\{ \left(\sum_a \hat{e}_a\hat{e}_a v^3_a\right) \left(\sum_{a'} \hat{e}_{a'}\hat{e}_{a'} v^3_{a'}\right)^* \right. \\ &\quad \left. + c.c. \right\} \left[\left(\sum_\sigma w^{3,3}_\sigma\right)^2 \frac{1}{3\xi^{3,3}}\right] : \ \frac{n\nabla\mathbf{u}}{3},\end{aligned} \tag{4.17}$$

from which the kinematic viscosity ν can be inferred. The expression in equation (4.17) is quite similar to the corresponding momentum tensor expression for the pure two-dimensional Navier-Stokes case, viz.

$$\pi^{(1)}_{NS} = \left\{ \left(\sum_a \hat{e}_a\hat{e}_a v^3_a\right) \left(\sum_{a'} \hat{e}_{a'}\hat{e}_{a'} v^3_{a'}\right)^* + c.c. \right\} \left[\frac{1}{\lambda_3}\right] : \ \frac{n\nabla\mathbf{u}}{3}, \tag{4.18}$$

where λ_3 is the eigenvalue of $\vec{v}^3$ for the Navier-Stokes case. λ_3 is given by

$$\frac{1}{\lambda_3(NS)} = -\frac{2}{n(1 - n/6)^3}. \tag{4.19}$$

The factor in equation (4.17) which corresponds to the factor $1/\lambda_3$ in equation (4.18) is, from Appendix D,

$$\left(\sum_\sigma w_\sigma^{3,3}\right)^2 \frac{1}{3\xi^{3,3}} = \frac{-2}{n\,(1-n/18)^{15}}, \tag{4.20}$$

Comparing equations (4.20) and (4.19), we see that in the limit of zero density ($n \to 0$), the viscosities for the 2D MHD CA are the same as those for the two-dimensional Navier-Stokes one, and differ by the factor $(1-n/18)^{15}/(1-n/6)^3$. In the important case of very low density,

$$\nu = \frac{1}{2n}, \ (n \to 0), \tag{4.21}$$

in agreement with the result of Wolfram [2].

4.2 Evaluation of the magnetic diffusivity

The coefficient $\rho^{c,\nu}$ relevant to the magnetic diffusivity is the second term on the right-hand side of equation (4.13):

$$\rho^{c,\nu}(\text{mag}) = \frac{1}{\xi^{c,\nu}} \sum_a \sum_\sigma \left(v_a^c w_\sigma^{c,\nu}\right)^* \frac{\hat{e}_a \sigma}{12} n \cdot \nabla A_z. \tag{4.22}$$

Since

$$\sum_a v_a^{c*} \hat{e}_a = 0 \qquad \text{for } c = 1,3,4,5, \tag{4.23}$$

and

$$\sum_\sigma \sigma w_\sigma^{2,\nu} = \sum_\sigma \sigma w_\sigma^{6,\nu} = 0 \qquad \text{for } \nu = 2,3, \tag{4.24}$$

only $\rho^{2,1}(\text{mag})$ and $\rho^{6,1}(\text{mag})$ have finite values. The first-order flux $\phi^{(1)}$ is

$$\phi^{(1)} = \sum_a \sum_\sigma \hat{e}_a \sigma \left(v_a^2 w_\sigma^{2,1} \rho^{2,1}(\text{mag}) + v_a^6 w_\sigma^{6,1} \rho^{6,1}(\text{mag})\right)$$

$$\left\{\left(\sum_a \hat{e}_a v_a^2\right)\left(\sum_{a'} \hat{e}_{a'} v_{a'}^2\right)^* + c.c.\right\} \cdot \left(\sum_\sigma \sigma w_\sigma^{2,1}\right)^2 \frac{n}{2\xi^{2,1}} \nabla A_z. \tag{4.25}$$

Using the formulas for $w^{2,1}$ and $\xi^{2,1}$ from Appendix D and the identity

$$\sum_{a=1}^{3} \hat{e}_a \hat{e}_a = \frac{3}{2}\,\mathbf{1},$$

we finally have, from equation (4.25),

$$\phi^{(1)} = \frac{1}{2} \frac{n \nabla A_z}{\left(3\tilde{f} + 9\tilde{f}^2\right)(1-n/18)^{16}}, \tag{4.26}$$

where $\tilde{f} \equiv n/(18-n)$. As $n \to 0$, the magnetic diffusivity tends toward

$$\eta = \frac{3}{n}, \quad (n \to 0), \tag{4.27}$$

which is to be compared with equation (4.21).

In summary, we have for the magnetic diffusivity η,

$$\eta = \frac{1}{2}\frac{1}{\left(3\tilde{f} + 9\tilde{f}^2\right)(1-n/18)^{16}}, \tag{4.28}$$

where $\tilde{f} = n/(18-n)$. For the kinematic viscosity, we have

$$\nu = \frac{1}{2n}\frac{1}{(1-n/18)^{15}}. \tag{4.29}$$

5. Discussion and concluding remarks

Equations (4.28) and (4.29) are the principal results of this paper. Their computational verification or disproof awaits the results of 2D MHD CA codes now in preparation. It is also worth a reminder that additional contributions to ν and η connected with finite discrete lattice size may contribute additive additional terms to equations (4.28) and (4.29) and remain to be evaluated.

It should also be noted that a definite magnetic Prandtl number $\nu/n \equiv P_{mag}$ ($\cong 1/6$, at low densities) is implied by equations (4.28) and (4.29). This parameter is one which, in computations, is one that it would be desirable to vary. It is clear that some variation of it should be possible by varying the fraction of $2R$, $2L$, and $3S$ collisions which are permitted to exchange σ-values among the particles. Just how wide the range in P_{mag} that this will permit is uncertain, and probably must be determined by computational practice.

The least-satisfying features of the 2D MHD CA remains the need for the microscopic velocity-flipping routine for the inclusion of the Lorentz force in equation (2.1). The code inevitably will be slowed down by its operation, and how serious a limitation is also something that can be decided only by computational practice.

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Appendix A. Enumeration of the terms in equation (4.1)

The terms on the right-hand side of equation (4.1), written out in detail, are $\Omega^{I}_{a,\sigma} + \Omega^{II}_{a,\sigma} + \Omega^{III}_{a,\sigma} + \Omega^{IV}_{a,\sigma}$, where

$$\Omega^{I}_{a,\pm1}/\left(\Pi_{a,\sigma}(1-f_{a,\sigma})\right) =$$

$$\frac{1}{2}\Big[\tilde{f}_{a+1,\pm1}\tilde{f}_{a+4,\pm1} + \tilde{f}_{a+2,\pm1}\tilde{f}_{a+5,\pm1}$$

$$+ \frac{1}{2}\left(\tilde{f}_{a+1,\pm1}\tilde{f}_{a+4,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+4,\pm1} + \tilde{f}_{a+2,\pm1}\tilde{f}_{a+5,0} + \tilde{f}_{a+2,0}\tilde{f}_{a+5,\pm1}\right)$$

$$+ \frac{1}{3}\Big(\tilde{f}_{a+1,1}\tilde{f}_{a+4,-1} + \tilde{f}_{a+1,0}\tilde{f}_{a+4,0} + \tilde{f}_{a+1,-1}\tilde{f}_{a+4,1}$$

$$+ \tilde{f}_{a+2,1}\tilde{f}_{a+5,-1} + \tilde{f}_{a+2,0}\tilde{f}_{a+5,0} + \tilde{f}_{a+2,-1}\tilde{f}_{a+5,1}\Big)\Big]. \tag{A.1}$$

$$\Omega^{I}_{a,0}/\left(\Pi_{a,\sigma}(1-f_{a,\sigma}\right) =$$

$$\frac{1}{2}\Big[\frac{1}{2}\left(\tilde{f}_{a+1,1}\tilde{f}_{a+4,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+4,1} + \tilde{f}_{a+2,1}\tilde{f}_{a+5,0} + \tilde{f}_{a+2,0}\tilde{f}_{a+5,1}\right)$$

$$+ \frac{1}{3}\Big(\tilde{f}_{a+1,1}\tilde{f}_{a+4,-1} + \tilde{f}_{a+1,0}\tilde{f}_{a+4,0} + \tilde{f}_{a+1,-1}\tilde{f}_{a+4,1} + \tilde{f}_{a+2,1}\tilde{f}_{a+5,-1}$$

$$+ \tilde{f}_{a+2,0}\tilde{f}_{a+5,0} + \tilde{f}_{a+2,-1}\tilde{f}_{a+5,1}\Big)$$

$$+ \frac{1}{2}\left(\tilde{f}_{a+1,0}\tilde{f}_{a+4,-1} + \tilde{f}_{a+1,-1}\tilde{f}_{a+4,0} + \tilde{f}_{a+2,0}\tilde{f}_{a+5,-1} + \tilde{f}_{a+2,-1}\tilde{f}_{a+5,0}\right)\Big]. \tag{A.2}$$

$$\Omega^{II}_{a,\sigma}/\left(\Pi_{a,\sigma}(1-f_{a,\sigma})\right) = -\tilde{f}_{a,\sigma}\left(\tilde{f}_{a+3,1} + \tilde{f}_{a+3,0} + \tilde{f}_{a+3,-1}\right). \tag{A.3}$$

$$\Omega^{III}_{a,\pm1}/\left(\Pi_{a,\sigma}(1-f_{a,\sigma})\right) = \tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,\pm1}$$

$$+ \frac{2}{3}\left(\tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,\pm1} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,\pm1}\right)$$

$$+ \frac{2}{3}\left(\tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,\mp1} + \tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,\mp1}\tilde{f}_{a+5,\pm1} + \tilde{f}_{a+1,\mp1}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,\pm1}\right)$$

$$+ \frac{1}{3}\left(\tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,0}\tilde{f}_{a+5,\pm1}\right)$$

$$+ \frac{1}{3}\Big(\tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,\mp1} + \tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,\mp1}\tilde{f}_{a+5,0}$$

$$+ \tilde{f}_{a+1,0}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,\mp1} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,\mp1}\tilde{f}_{a+5,\pm1}$$

$$+ \tilde{f}_{a+1,\mp1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,\pm1} + \tilde{f}_{a+1,\mp1}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,0}\Big)$$

$$+ \frac{1}{3}\Big(\tilde{f}_{a+1,\mp1}\tilde{f}_{a+3,\mp1}\tilde{f}_{a+5,\pm1} + \tilde{f}_{a+1,\mp1}\tilde{f}_{a+3,\pm1}\tilde{f}_{a+5,\mp1}$$

$$+ \tilde{f}_{a+1,\pm1}\tilde{f}_{a+3,\mp1}\tilde{f}_{a+5,\mp1}\Big) . \tag{A.4}$$

$$\Omega^{III}_{a,0} / \left(\Pi_{a,\sigma}(1 - f_{a,\sigma})\right)$$

$$= \frac{1}{3}\Big(\tilde{f}_{a+1,1}\tilde{f}_{a+3,1}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,1} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,1}\tilde{f}_{a+5,1}$$

$$+ \frac{2}{3}\Big(\tilde{f}_{a+1,1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,1}\tilde{f}_{a+5,0}$$

$$+ \tilde{f}_{a+1,0}\tilde{f}_{a+3,0}\tilde{f}_{a+5,1}\Big)$$

$$+ \frac{1}{3}\Big(\tilde{f}_{a+1,1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,-1} + \tilde{f}_{a+1,1}\tilde{f}_{a+3,-1}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,1}\tilde{f}_{a+5,-1}$$

$$+ \tilde{f}_{a+1,0}\tilde{f}_{a+3,-1}\tilde{f}_{a+5,1}$$

$$+ \tilde{f}_{a+1,-1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,1} + \tilde{f}_{a+1,-1}\tilde{f}_{a+3,1}\tilde{f}_{a+5,0}\Big) + \tilde{f}_{a+1,0}\tilde{f}_{a+3,0}\tilde{f}_{a+5,0}$$

$$+ \frac{2}{3}\Big(\tilde{f}_{a+1,-1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,-1}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,0}\tilde{f}_{a+5,-1}\Big)$$

$$+ \frac{1}{3}\Big(\tilde{f}_{a+1,-1}\tilde{f}_{a+3,-1}\tilde{f}_{a+5,0} + \tilde{f}_{a+1,-1}\tilde{f}_{a+3,0}\tilde{f}_{a+5,-1} + \tilde{f}_{a+1,0}\tilde{f}_{a+3,-1}\tilde{f}_{a+5,-1}\Big) . \tag{A.5}$$

$$\Omega^{IV}_{a,\sigma} / \left(\Pi_{a,\sigma}(1 - f_{a,\sigma})\right) = -\tilde{f}_{a,\sigma}\Big(\tilde{f}_{a+2,1}\tilde{f}_{a+4,1} + \tilde{f}_{a+2,1}\tilde{f}_{a+4,0} + \tilde{f}_{a+2,0}\tilde{f}_{a+4,1}$$

$$+ \tilde{f}_{a+2,0}\tilde{f}_{a+4,0} + \tilde{f}_{a+2,1}\tilde{f}_{a+4,-1} + \tilde{f}_{a+2,-1}\tilde{f}_{a+4,1}$$

$$+ \tilde{f}_{a+2,-1}\tilde{f}_{a+4,0} + \tilde{f}_{a+2,0}\tilde{f}_{a+4,-1} + \tilde{f}_{a+2,-1}\tilde{f}_{a+4,-1}\Big) . \tag{A.6}$$

Appendix B. Explicit form of the matrices $\vec{\vec{\omega}}^{(j)}$ and $\tau^{(j)}$ of equation (4.2)

The fluid velocity u and the magnetic vector potential A_z are both considered to be small compared to unity. For purposes of inverting $\vec{\vec{C}}^{(0)}$, they may be dropped, leaving $f^{(0)}_{a,\sigma} = n/18 +$ (higher-order terms). Calling $n/18 \equiv f$, and $\tilde{f} = f/(1-f)$, the explicit forms are:

$$\vec{\vec{\omega}}^{(1)} = (1-f)^{16}\,\mathrm{circ}\,[0,\ 1,\ 1,\ 0,\ 1,\ 1](\tilde{f}/2), \tag{B.1}$$

(where "circ" means a 6×6 circulant matrix),

$$\tau^{(1)} = \begin{pmatrix} 11/6 & 5/6 & 2/6 \\ 5/6 & 8/6 & 5/6 \\ 2/6 & 5/6 & 11/6 \end{pmatrix}, \tag{B.2}$$

$$\vec{\vec{\omega}}^{(2)} = (1-f)^{16}\,\mathrm{circ}\,[0,\ 1,\ 0,\ 1,\ 0,\ 1]\tilde{f}^2 \tag{B.3}$$

$$\tau^{(2)} = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{pmatrix}, \tag{B.4}$$

$$\vec{\vec{\omega}}^{(3)} = -(1-f)^{16}\,\mathrm{circ}\,[9\tilde{f}^2 + 3\tilde{f},\ 0,\ 0,\ 0,\ 0,\ 0], \tag{B.5}$$

$$\tau^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.6}$$

$$\vec{\vec{\omega}}^{(4)} = -(1-f)^{16}\,\mathrm{circ}\,[0,\ 0,\ 3\tilde{f}^2,\ \tilde{f},\ 3\tilde{f}^2,\ 0], \tag{B.7}$$

$$\tau^{(4)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{B.8}$$

Note that only $\tau^{(1)}$, among the τ's, is anything other than a circulant matrix.

Appendix C. Eigenvectors $\vec{v}^c$ and eigenvalues $\lambda^{c(i)}$ for the $\vec{\vec{\omega}}^{(i)}$ matrices, equation (4.4)

The six eigenvectors and corresponding eigenvalues are [2,16]:

$$\vec{v}^1 = (1,\ 1,\ 1,\ 1,\ 1,\ 1)/\sqrt{6}$$

$$\vec{v}^2 = (1,\ \sigma_0,\ -\sigma_0^*,\ -1,\ -\sigma_0,\ \sigma_0^*)/\sqrt{6}$$

$$\vec{v}^3 = (1,\ -\sigma_0*,\ -\sigma_0,\ 1,\ -\sigma_0^*,\ -\sigma_0)/\sqrt{6}$$

$$\vec{v}^4 = (1,\ -1\ ,\ 1,\ -1,\ 1,\ -1)/\sqrt{6}$$

$$\vec{v}^5 = (1,\ -\sigma_0,\ -\sigma_0^*,\ 1,\ -\sigma_0,\ -\sigma_0^*)/\sqrt{6}$$

$$\vec{v}^6 = (1,\ \sigma_0^*,\ -\sigma_0,\ -1,\ ,-\sigma_0^*,\ \sigma_0)/\sqrt{6}, \tag{C.1}$$

where $\sigma_0 = (1 + i\sqrt{3})/2$.

The corresponding eigenvalues are

$$\begin{aligned}
\lambda^{1(1)} &= 2\tilde{f}(1-f)^{16} \\
\lambda^{2(1)} &= 0 \\
\lambda^{3(1)} &= -\tilde{f}(1-f)^{16} \\
\lambda^{4(1)} &= 0 \\
\lambda^{5(1)} &= -\tilde{f}(1-f)^{16} \\
\lambda^{6(1)} &= 0,
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\lambda^{1(2)} &= 3\tilde{f}^2(1-f)^{16} \\
\lambda^{2(2)} &= 0 \\
\lambda^{3(2)} &= 0 \\
\lambda^{4(2)} &= -3\tilde{f}^2(1-f)^{16} \\
\lambda^{5(2)} &= 0 \\
\lambda^{6(2)} &= 0,
\end{aligned} \tag{C.3}$$

$$\lambda^{c(3)} = -(3\tilde{f} + 9\tilde{f}^2)(1-f)^{16}, \text{ for arbitrary c.}$$

$$\begin{aligned}
\lambda^{1(4)} &= -(\tilde{f} + 6\tilde{f}^2)(1-f)^{16} \\
\lambda^{2(4)} &= (\tilde{f} + 3\tilde{f}^2)(1-f)^{16} \\
\lambda^{3(4)} &= -(\tilde{f} - 3\tilde{f}^2)(1-f)^{16} \\
\lambda^{4(4)} &= (\tilde{f} - 6\tilde{f}^2)(1-f)^{16} \\
\lambda^{5(4)} &= -(\tilde{f} - 3\tilde{f}^2)(1-f)^{16} \\
\lambda^{6(4)} &= (\tilde{f} + 3\tilde{f}^2)(1-f)^{16}.
\end{aligned} \tag{C.4}$$

As for f and $\tilde{f}$, see Appendix B; again, $f = n/18$, $\tilde{f} = f/(1-f)$, for purposes of inverting $\vec{\vec{C}}^{(0)}$.

Appendix D. The matrix τ^c of equation (4.7) and its eigenvectors $w^{c,\nu}$ and eigenvalues $\xi^{c,\nu}$

Explicitly, τ^c is a symmetric matrix:

$$\tau^c = \begin{pmatrix} a^c & d^c & e^c \\ d^c & b^c & d^c \\ e^c & d^c & a^c \end{pmatrix}, \tag{D.1}$$

where

$$\begin{aligned} a^c &= \frac{11}{6}\lambda^{c(1)} + 5\lambda^{c(2)} + \lambda^{c(3)} + \lambda^{c(4)} \\ b^c &= \frac{4}{3}\lambda^{c(1)} + 5\lambda^{c(2)} + \lambda^{c(3)} + \lambda^{c(4)} \\ d^c &= \frac{5}{6}\lambda^{c(1)} + 2\lambda^{c(2)} + \lambda^{c(4)} \\ e^c &= \frac{1}{3}\lambda^{c(1)} + 2\lambda^{c(2)} + \lambda^{c(4)}. \end{aligned} \tag{D.2}$$

The $\lambda^{c(i)}$ are given in Appendix C.

The first eigenvector and eigenvalue are

$$\begin{aligned} w^{c,1} &= \frac{1}{\sqrt{2}}(1,\ 0,\ -1) \\ \xi^{c,1} &= \frac{3}{2}\lambda^{c(1)} + 3\lambda^{c(2)} + \lambda^{c(3)}, \end{aligned} \tag{D.3}$$

and in particular,

$$\xi^{2,1} = -(3\tilde{f} + 9\tilde{f}^2)(1-f)^{16}. \tag{D.4}$$

The second and third eigenvectors are not simple; but those for $c = 3$ and $c = 5$, which are identical and are required to evaluate the viscosity, are

$$\begin{aligned} w^{3,2} = w^{5,2} &= \frac{1}{\sqrt{6}}(1,\ -2,\ 1) \\ w^{3,3} = w^{5,3} &= \frac{1}{\sqrt{3}}(1,\ 1,\ 1), \end{aligned} \tag{D.5}$$

and the eigenvalues are

$$\begin{aligned} \xi^{3,2} = \xi^{5,3} &= b^3 - d^3 = \left(-\frac{7}{2}\tilde{f} - 9\tilde{f}^2\right)(1-f)^{16} \\ \xi^{3,3} = \xi^{5,3} &= b^3 + 2d^3 = -9\tilde{f}(1-f)^{16}. \end{aligned} \tag{D.6}$$

Again, $f = n/18$, and $\tilde{f} = f/(1-f)$ (see Appendix B).

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