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Quantum Fractals

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Abstract. The fractal nature of quantum paths contributing to Feynman path integral formulation of quantum mechanics is investigated. A computer simulation of both one- and two-dimensional quantum harmonic oscillators yields results in agreement with rigorous results on the Hausdorff-Besicovitch dimension for Brownian motion.

1. Introduction

It is well known that classical mechanics can be reformulated in terms of a minimum principle. The Euler-Lagrange equations of motion follow from demanding that the action

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}) dt$$

be stationary with respect to variations in the path $x(t)$ taken between $x(t_i) = x_i$ and $x(t_f) = x_f$. An alternative formulation of quantum mechanics, due to Feynman [1], is in terms of a kernel, $F(x_f, t_f; x_i, t_i)$, defined by

$$\Psi(x_f, t_f) = \int dx_i F(x_f, t_f; x_i, t_i)\, \Psi\,(x_i, t_i)$$

The kernel is expressed as a "path integral": the path integral sums over all possible trajectories $x(t)$ from $x(t_i) = x_i$ to $x(t_f) = x_f$ with an amplitude which depends on the classical action for that path. Formally, this is written as

$$F(x_f, t_f; x_i, t_i) = (\text{const}) \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t) \exp[\frac{i}{\hbar} S[x(t)]]$$

where the symbol $\mathcal{D}x(t)$ is Feynman's famous "sum over all paths", and the overall constant does not concern us here. The sum over all paths may be defined more precisely by introducing a time lattice and dividing up the time interval, $t_f - t_i$, into equal time slices, ε apart, and integrating over all x_n at each time slice t_n.

$$\int \mathcal{D}x(t) \to \Pi_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n$$

where $N = (t_f - t_i)/\varepsilon$. The limit as $N \to \infty$ defines the path integral. More details on path integrals and the relationship to the Trotter product formula are to be found in the book by Schulman [2].

In this paper, we are more concerned with the question of which paths are most important in this path integral. The classical path corresponds to a minimum of the action, and since δS is zero for small variations $\delta x(t)$ about the classical trajectory $x_c(t)$, these paths interfere constructively and dominate the path integral as $\hbar \to 0$. However, as observed by Feynman and Hibbs (see figure 1 taken from [1]), quantum mechanical paths are very irregular. Furthermore, as indicated in the figure, when observed at a finer resolution, they appear just as irregular as on the coarser resolution. In fact, the paths are everywhere continuous but nowhere differentiable. This is the characteristic behavior of a fractal [3] and mathematicians, in particular Hausdorff and Besicovitch [4,5], have investigated the properties of such curves extensively. The classical example of a fractal is the so-called Koch curve, which is constructed by the sequence of steps shown in figure 2. At each increase in resolution by a factor of 3, the length of the curve increases by 4/3. It is clear that the measured length of this curve will depend on the resolution of our measuring instruments: the normal definition of length is therefore not very helpful for such objects. A modified definition of length for such self-similar curves, the so-called fractal length L, has been introduced by Mandelbrot [3] as

$$L = \ell(\Delta x)^{D-1}$$

where ℓ is the usual length measured when resolution is Δx and D, the fractal dimension, is a number chosen so that L will be independent of Δx in the limit $\Delta x \to 0$. (We will give a mathematically more precise definition of fractal dimension later in the paper.) Notice that for differentiable curves whose length, ℓ, is independent of Δx, D reduces to one, the usual topological dimension of a line. For the Koch curve, however, for two successive steps $\ell_2 = 4/3\ell_1$ and $\Delta x_2 = \frac{1}{3}(\Delta x_1)$. Thus, demanding that the fractal length L be independent of Δx yields the result $D = \ell n 4/\ell n 3 \simeq 1.2618$.

Having observed that paths contributing to the path integral are continuous but non-differentiable, one is naturally led to ask whether one can determine a fractal dimension for these paths. It is this question which we set out to answer in this paper, and along the way, we clarify the connection between quantum mechanics and Brownian motion, and between the fractal dimension of Mandelbrot and the Hausdorff-Besicovitch dimension of mathematicians.

The paper is constructed as follows. In the next section, we discuss Monte-Carlo simulations of quantum mechanics and explicitly analyze the problem of one- and two-dimensional harmonic oscillators using a euclidean version of the path integral. Our data allows us to determine the relevant

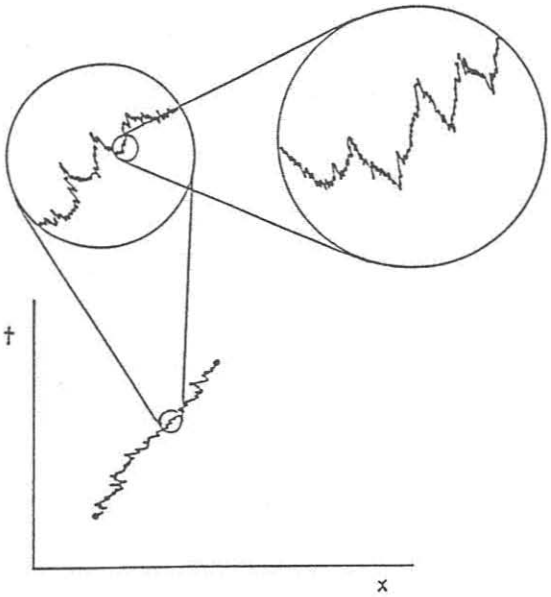


Figure 1: Typical quantum path.

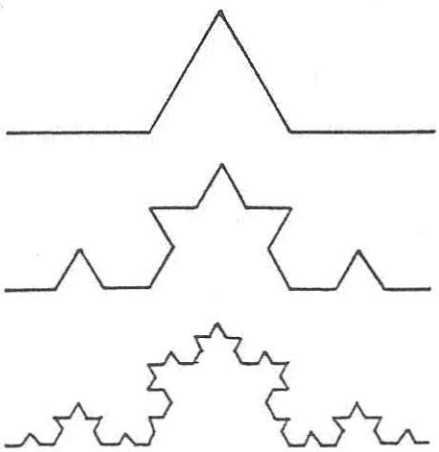


Figure 2: Construction of the Koch curve.

fractal dimension for the quantum paths in these cases, and we contrast this with the prediction of Abbot and Wise [6]. Section 3 contains a heuristic discussion of the mathematics of fractals and a review of Hausdorff and Besicovitch's analysis and definition of fractal dimension. The work of Levy and Taylor [7–9] on Brownian paths is discussed along with the connection between these rigorous mathematical statements and Mandelbrot's heuristic definition of D. The relevance of these analyses to our quantum example will then be clear. Our conclusions are briefly summarized in section 4.

2. Monte-Carlo simulations of quantum harmonic oscillator

In this section, we present the results of Monte-Carlo simulations of Feynman's path integral formulation of the quantum harmonic oscillator problem. In the imaginary time formalism, the Feynman path integral is mathematically equivalent to a partition function of a statistical mechanical system with only nearest-neighbor interactions. As shown by Creutz and Freedman [10], this euclidean version of the harmonic oscillator can provide information on the ground and first excited states of this problem. Our interest in such simulations is rather different: we wish to examine the fractal nature of the trajectories that dominate the path integral.

The formalism is by now fairly standard [10]. In euclidean time, the kernel becomes

$$F \sim \int \mathcal{D}x \, e^{-S[x]/\hbar}$$

with the euclidean action given by

$$S = \int_0^T d\tau [\frac{1}{2}(\frac{dx}{d\tau})^2 + V(x)]$$

where $\tau = it$ (t is real time) and $\int \mathcal{D}x$ denotes integration over all functions $x(\tau)$ obeying the boundary conditions:

$$\begin{aligned} x(0) &= x_i \\ x(\tau) &= x_f \end{aligned}$$

To define the "sum over all paths" precisely, we introduce a time lattice as discussed in the introduction. The action for a discrete time lattice is just

$$S = \sum_{j=1}^{N} a[\frac{1}{2} m_0(\frac{x_{j-1} - x_j}{a})^2 + V(x_j)]$$

where $a = i\varepsilon$ is the euclidean lattice spacing.

To perform the sum over paths, we use the standard Monte-Carlo method due to Metropolis et. al. [11]. To make connection with continuum physics, the lattice size a must be sufficiently small and Na sufficiently large to isolate the ground-state properties of the theory. It is straightforward to reproduce the results of reference [10] for the one-dimensional harmonic oscillator with

$$V(x) = \frac{1}{2}\,\mu x^2.$$

To investigate the fractal character of the dominant paths contributing to the path integral, we need to examine the variation in length of these paths as we vary the "resolution" of each simulation. Thus, for fixed $t = Na$, we varied N and a and calculated the average path lengths as a function of $a = \Delta\tau$. Then, according to Mandelbrot's formula

$$L = \ell(\Delta x)^{D-1}$$

a plot of $\ell n(\ell)$ versus $\ell n(\Delta\tau)$ has a slope of $(1 - D)$, enabling Mandelbrot's fractal dimension D to be determined.

The numerical simulations were carried out on a number of ICL computers: the PERQ and the 2970 at Southampton and the Distributed Array Processor (DAP) at QMC London. A thermalization time of 50 to 60 Monte-Carlo sweeps was allowed in most simulations. Values of "a" ranged from 0.02 to 0.1. For each value of a, the optimum value of the Monte-Carlo change, Δx_{opt}, was found by finding the value which resulted in the fastest thermalization and the smallest statistical error. We found that Δx_{opt} equal to $(a)^{1/2}$ was a good approximation in general. For each data point, we generated about 8000 sweeps; however, to minimize the effects of correlations in the Monte-Carlo data, measurements were carried out only on every tenth sweep. To estimate the correlation in the remaining data, the method of Daniell, Hey, and Mandula [12] was used.

Our results for the one- and two-dimensional harmonic oscillator are shown in figures 3 and 4 respectively. In both cases, we see that the so-called fractal dimension D is approximately 1.5. This is in contrast to the fractal dimension $D = 2$ suggested by Abbot and Wise [6] in the context of quantum trajectories.

How can our results be understood? For Brownian motion, which as we will see, is closely related to this quantum mechanical problem, it can be rigorously shown [7–9] that the fractal dimension of a Brownian graph is 1.5 for one-dimensional Brownian motion but 2 for two- or three-dimensional motion. Our results therefore seem surprising, since we do not expect the presence of a smooth potential $V(x)$ in the quantum problem to modify these conclusions from Brownian motion. We were therefore led to examine in more detail the connection between Mandelbrot's heuristic definition of D and the rigorous mathematical results available on Brownian motion and their relevance to the quantum mechanical problem.

3. The Hausdorff-Besicovitch dimension

The procedure to be described was first set down by Hausdorff and Besicovitch [4,5] and is applicable to all kinds of geometrical objects. Since we are primarily interested in Feynman paths, we will restrict our discussion to objects of topological dimension one: generalization to higher topological dimensions is straightforward.

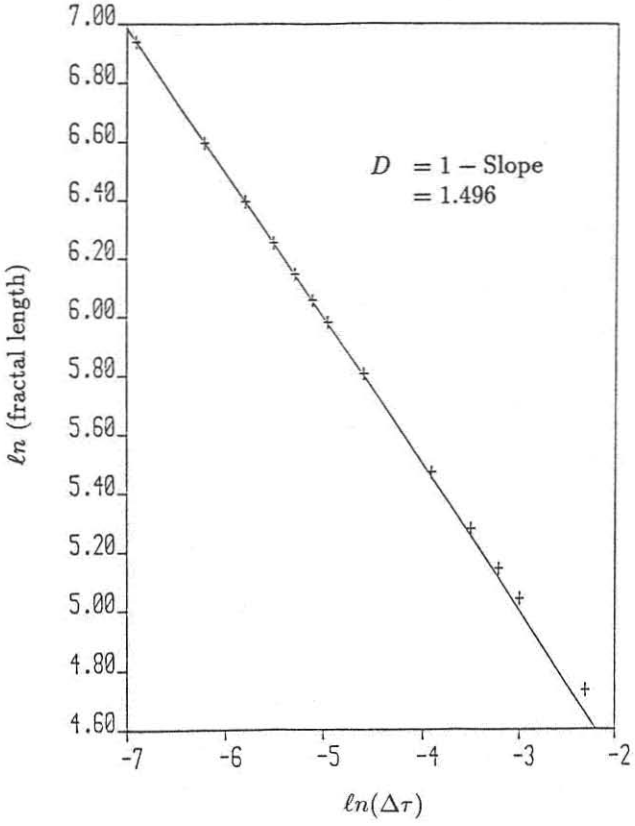


Figure 3: Quantum path in one space dimension.

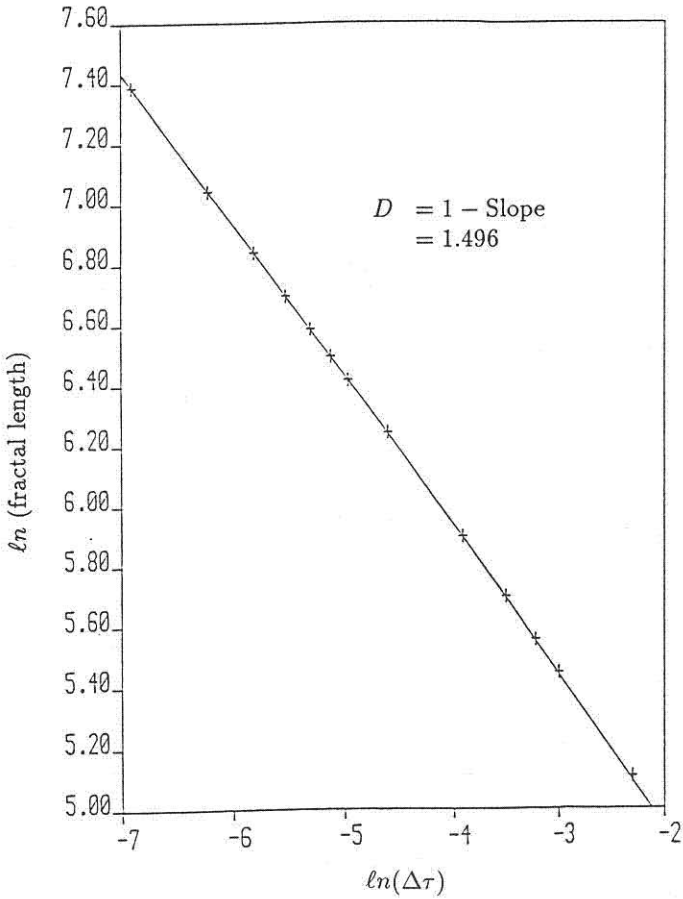


Figure 4: Quantum path in two space dimensions.

To determine the Hausdorff-Besicovitch (HB) dimension, we first cover the curve completely with a finite set of spheres of various radii "ρ_i". These spheres may overlap. For each of these spheres, we calculate the quantity

$$h(\rho_i) = (\rho_i)^d$$

where d is a real number whose value is arbitrary for the moment. We then sum the results:

$$\sigma\{\rho_i\} = \sum_i h(\rho_i)$$

Now consider all such coverings in which no spheres are used with radii greater than some radius "ρ". Since $\sigma\{\rho_i\}$ is greater than zero for all such coverings, there must be a greatest lower bound (infinum) S_ρ for $\sigma\{\rho_i\}$.

$$S_\rho = \mathrm{Inf}_{(\rho_i<\rho)}\, \sigma\{\rho_i\}$$

Finally, let $\rho \to 0$; since this simply reduces the set of possible coverings allowed (from which we are finding the "smallest" $\sigma\{\rho_i\}$), S_ρ can only increase or stay constant. It follows that S_ρ tends to some limit S_0 which is either infinite or a finite non-negative number.

The value of S_0 depends on the value d used in the construction of $h(\rho_i)$. The essential step of HB was the proof that there exists a unique number D such that:

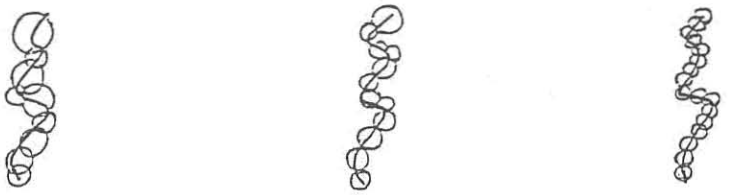
$$\text{For any } d > D \qquad S_0 = \lim_{(\rho\to 0)} S_\rho = 0$$

$$\text{For any } d < D \qquad S_0 = \lim_{(\rho\to 0)} S_\rho = \infty$$

This D is called the Hausdorff-Besicovitch dimension.

For simple objects, D will be equal to the topological dimension (e.g. for a plane surface $D = 2$, $h(\rho) = \rho^2$ and the spheres indeed measure area), but this need not be the case for geometrical objects with detail on every small scale and indeed D may turn out to be non-integral. Such objects have been called fractals by Mandelbrot [3]. Note that we have said nothing about the limit of S_ρ at $d = D$. This may be zero, finite and positive, or infinite. This situation is illustrated in figure 5. In general, for a fractal, using $h(\rho) = \rho^D$ will lead to $S_0 = 0$ or $S_0 = \infty$; some more subtle function $h(\rho)$ is needed to obtain a non-trivial value. For example, the path left by Brownian motion in two or more dimensions has HB dimension $D = 2$, but the use of $h(\rho) = \rho^2$ leads to $S_0 = 0$. In three or more spatial dimensions, however, use of $h(\rho) = \rho^2 \log \log(1/\rho)$ leads to a finite positive answer for S_0 [6]. The more subtle function does not change the power law behavior of $h(\rho)$. For example, in this case we have

$$\left.\begin{array}{ll} h(\rho)/\rho^\lambda & \to 0 \text{ for } \lambda < 2 \\ & \to \infty \text{ for } \lambda > 2 \end{array}\right\} \begin{array}{c}\text{as} \\ \rho \to 0.\end{array}$$



Curve and simple covering | Infinum covering $\rho_i < \rho$ | $\rho \to 0$

Figure 5: Curve and simple coverings.

In a sense, the form of $h(\rho)$ is indicating that the HB dimension is infinitesimally less than two [3] since in fact:

$$h(\rho)/\rho^{\lambda} \to \infty \text{ for } \lambda \geq 2 \qquad \text{as } \rho \to 0.$$

It should therefore be clear that $h(\rho)$ need only be known up to a power in ρ in order to determine the HB dimension. The function $h(\rho)$ that yields a finite positive result for S_0 at $d = D$ is known as the intrinsic (see figure 6). The limit S_0 is known as the measure and we see here that it depends on the function $h(\rho)$.

Some simple properties of the HB dimension are intuitively obvious (and easily proved):

1. $D \geq$ (topological dimension of the space), e.g. for our quantum paths $D \geq 1$.

2. $D \leq$ (topological dimension of the space in which the curve is embedded).

3. The HB dimension D_p of the projection of a curve down on to a subspace satisfies $D_p \leq D$ (where D is the HB dimension of the original curve).

We now discuss the connection of the HB dimension with Mandelbrot's heuristic definition of a fractal dimension. Mandelbrot defines D by the formula

$$L = \ell(a)^{D-1}$$

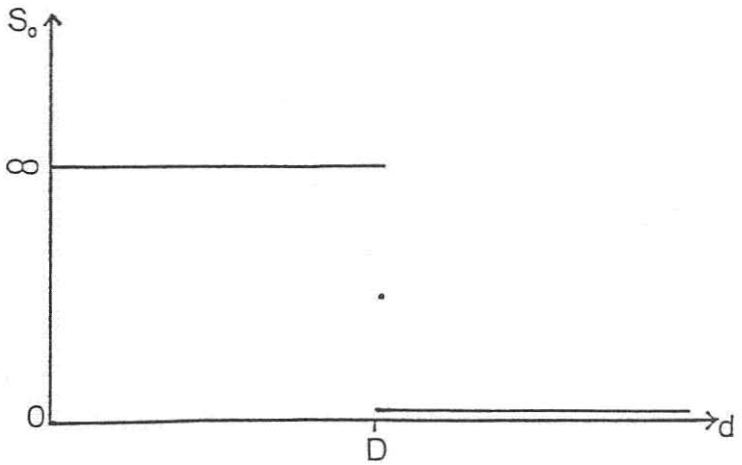


Figure 6: Graph of $S_0 = \lim_{\rho \to 0} S_\rho$ as a function of d.

where L is a quantity independent of the "resolution" a, and ℓ is the length of the curve at the resolution "a". This formula strictly only holds for statistically self-similar curves—curves whose behavior looks exactly the same (or statistically the same) on every scale.

Suppose that instead of covering the curve with spheres of varying radii $\rho_i \leq \rho$, we cover the curve with spheres of all the same radius "a". We can still consider all possible such coverings and form the infinum value of the quantity, $\sum h(a)$, where the sum is over all spheres in the covering. In this case, therefore, we can define a length L_a by

$$\begin{aligned} L_a &= \text{Inf} \sum h(a) \\ &= Nh(a) \\ &= Na^d \end{aligned}$$

where N is the minimum number of spheres of radius a needed to cover the curve. Now consider the limit L_0 of (Na^d) as $a \to 0$. Since

$$\text{Inf}_{(\rho_i = a)} \sum_i h(\rho_i) \geq \text{Inf}_{(\rho_i \leq a)} \sum_i h(\rho_i)$$

we have

$$L_0 \geq \lim_{(a \to 0)} S_a = S_0$$

Moreover, it is clear that there must be some value D' such that

$$\begin{aligned} &\text{for all} \quad d > D', L_0 = 0 \qquad \text{while} \\ &\text{for all} \quad d < D', L_0 = \infty. \end{aligned}$$

The inequality above then implies that $D' \geq D$.

We may reformulate this definition of length and fractal dimension D in terms of "resolutions". If "a" is the resolution, then, using this infinum covering, we would call the length of the curve $\ell = Na$. We may then rewrite our expression for L_a in terms of ℓ and a:

$$L_a = \ell\,(a)^{d-1}$$

If, further, we only determine this length ℓ up to some power of "a", then there will be some value of "d" for which L_a will not depend on "a" as $a \to 0$. This is $d = D' = D$.

Hence we arrive at

$$L = \ell(a)^{D-1} \quad (L \text{ independent of } a)$$

Although this formula was originally introduced by Mandelbrot for self-similar curves [3], it should be clear from our derivation that, provided we can determine the appropriate "a" for a given approximation to a fractal, this formula may be applied to fractals that are not self-similar.

Note that, except in very simple cases, no importance can be attached to the value of L: the formula being derived from approximations in the covering and "ℓ" being determined only up to a power law in "a". The determination of the HB dimension is reduced to determining the behavior of the length "ℓ" as a power law in the resolution according to $\ell \propto a^{1-D}$.

4. The fractal dimension of paths associated with the path integral

As stated before, the most important paths contributing to the path integral are paths lying closely around the classical path which are highly irregular and non-differentiable on all scales. The natural approximation to these paths is that obtained by chopping up the path integral into time slices $\Delta t = \varepsilon$. Consider the action

$$S[\underline{x}] = \int (\frac{m\dot{\underline{x}}^2}{2} - V(\underline{x}))\, dt,$$

for K space dimensions $\underline{x} = x_1, \ldots, x_K$). Rotating into euclidean space, the path integral is

$$\int \mathcal{D}(x(t)) e^{\frac{i}{\hbar} S[x(t)]} \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} d^K x_n \; e^{\frac{-1}{\hbar} \sum_{n=1}^{N-1} \{\frac{m}{2\varepsilon}(\underline{x}_{n+1} - \underline{x}_n)^2 + V(\underline{x}_n)\varepsilon\}}$$

The jump, Δx, in the x position for a time increment $\Delta t = \varepsilon$ is given by

$$\begin{aligned}(\Delta x)^2 &= \langle (\underline{x}_{m+1} - \underline{x}_m)^2 \rangle \\ &= \frac{\prod_{n=1}^{N-1} \int_{-\infty}^{\infty} d^K x_n e^{-\frac{1}{\hbar} \sum \{\frac{m}{2\varepsilon}(\underline{x}_{n+1} - \underline{x}_n)^2 + V(\underline{x}_n)\varepsilon\}} (\underline{x}_{m+1} - \underline{x}_m)^2}{\prod_{n=1}^{N-1} \int_{-\infty}^{\infty} d^K x_n e^{-\frac{1}{\hbar} \sum \{\frac{m}{2\varepsilon}(\underline{x}_{n+1} - \underline{x}_n)^2 + V(\underline{x}_n)\varepsilon\}}}\end{aligned}$$

Clearly, as $\varepsilon \to 0$, the potential term becomes unimportant and the integrals factorize. Hence, we obtain (for sufficiently small ε) the well-known result that

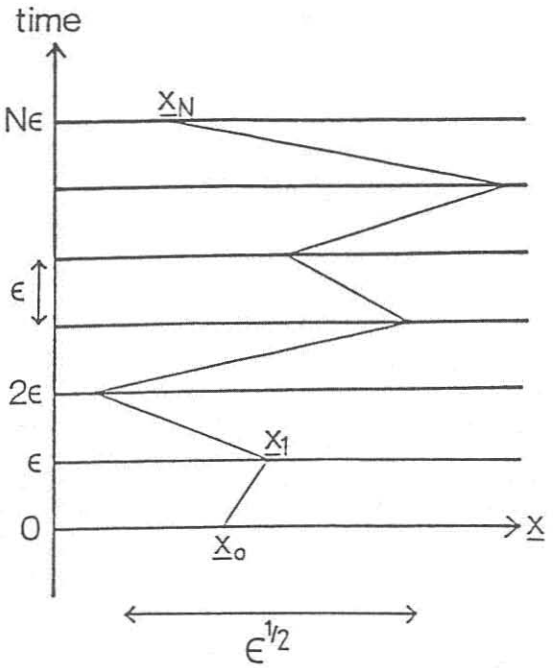


Figure 7: The graph.

$$\Delta x \sim \varepsilon^{1/2}$$

This formula is sufficient to determine the fractal dimension of both the *graph*, a plot of $\underline{x}(t)$ against time (see figure 7), and the *path*, the trace left in K-space as time evolves (see figure 8). Notice that the length of a link in both the graph and the path is $\varepsilon^{1/2}$ (since $\varepsilon \ll \varepsilon^{1/2}$ for $\varepsilon \to 0$). This means that the total length of both graph and path for time slice $\Delta t = \varepsilon$ may be defined as $\ell = N\,\varepsilon^{1/2} = (T/\varepsilon)\,\varepsilon^{1/2} = T\,\varepsilon^{-1/2}$ (where T is the total time).

In order to use the final formula of the previous section, we must now determine the resolution "a". It is important to realize that the resolution is not necessarily $a = \Delta t = \varepsilon$: it should be clear from the analysis in the previous section that the resolution "a" has nothing to do with time slices, *a priori*, but is equal to the radius of spheres that define the size of the detail in the diagram. To be more precise, we can give the following definition: let the resolution be $a \propto \varepsilon^p$ (where p is some power to be determined). This is the radius of a sphere that on the average, as $\varepsilon \to 0$, covers two or more lines in the diagram, but not infinitely many. This is illustrated for the *path* (in $K \geq 2$ dimensions) in figure 9 with step size $\Delta x \sim \varepsilon^{1/2}$. Any sphere with radius ε^p with $p < 1/2$ would become infinitely large (as $\varepsilon \to 0$) compared with the $\varepsilon^{1/2}$ step size (figure 9a). Conversely, any sphere with $p > 1/2$

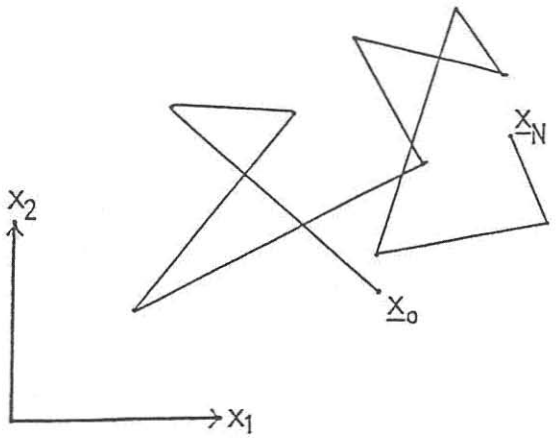


Figure 8: The path.

will become infinitely small compared with the steps $\varepsilon^{1/2}$ in figure 9b. It is therefore clear that the appropriate resolution is for the radius of the sphere to be $a = \varepsilon^{1/2}$. Hence,

$$\ell \propto \varepsilon^{-1/2} \propto a^{-1}$$

and from the final formula of the last section, $D = 2$.

Note that since the length of the links in the *graph* is also $\varepsilon^{1/2}$, we may ignore the time displacements of ε and obtain a diagram similar to that shown in figure 9 (for $K \geq 2$ dimensions). Hence, for the graph we also have $D = 2$.

The case of one spatial dimension ($K = 1$) must, however, be treated separately. In this case, the diagram in figure 9 is no longer appropriate for both the graph and the path. For the path, two links can overlap, whereas in the graph, the two links are spread ε apart. Using properties (1) and (2) of the HB dimension (given in the previous section), we have immediately the result that $D = 1$ for the path. For the graph, the separation of ε between overlapping links implies that we must take $a \propto \varepsilon$. Hence,

$$\ell \propto \varepsilon^{-1/2} = a^{-1/2}$$

and in this case, $D = 1.5$. It is important to note that in this case, the fractal is *not* self-similar: as we increase our resolution (i.e. decrease $a = \varepsilon$), the jumps in the curve vary ever more widely since $(\Delta x/a) \to \infty$ as $a \to 0$. Thus, this fractal does not have the same behavior on all scales. Nevertheless, from our derivation of Mandelbrot's formula, we see that it is still possible to determine a fractal dimension.

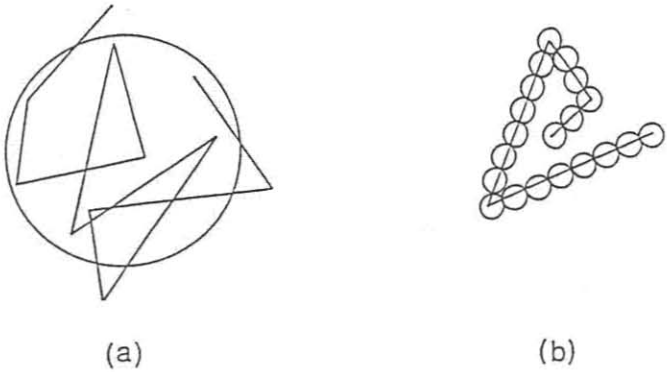


Figure 9: Different covering scales for Brownian paths.

K	D_{path}	D_{graph}
1	1	1.5
≥ 2	2	2

Table 1: HB dimensions of paths and graphs for different K-space dimensions.

These results are summarized in table 1. Notice that $D_{\text{graph}} \geq D_{\text{path}}$, consistent with property (3) of the previous section. (The path is obtained from the graph by projecting down the time axis.) It is worth stressing that the formula $\ell \propto \varepsilon^{-1/2}$ is true for *all* entries in the table; the differing HB dimensions arise from different choices of resolution "a".

Finally, in this section we should comment on the work of Taylor [8,9]. Taylor studied extensively the fractal nature of Brownian paths. These paths are a set of functions $\{\underline{x}(t)\}$ on which a certain probability measure is defined [8]. One of the definitions required is that the probability of $x(t_1) \in [\alpha_1, \beta_1]$ and $x(t_2) \in [\alpha_2, \beta_2]$ and ... and $x(t_m) \in [\alpha_m, \beta_m]$ (in $K = 1$ dimensions) is

$$\mu = (2\pi)^{\frac{m}{2}} [t_1 \Pi_2^m (t_r - t_{r-1})]^{-1/2} \int_{\alpha_1}^{\beta_1} dx_1 \int_{\alpha_2}^{\beta_2} dx_2 \ldots$$
$$\int_{\alpha_m}^{\beta_m} dx_m \; e^{-[\frac{x_1^2}{2t_1} + \sum_2^m \frac{(x_r - x_{r-1})^2}{2(t_r - t_{r-1})}]}$$

Since the potential played no role in our analysis of the fractal dimension of the quantum mechanical path integral, it is clear from the similarity of that analysis and this definition that Taylor's Brownian motion will have the same fractal nature as our quantum mechanical paths. Indeed, Taylor proved that Brownian paths in K-space ($K \geq 2$) have fractal dimension $D = 2$ [8]

and that the graph of a one-dimensional Brownian path has $D = 1.5$ [9]. In addition, he obtained information regarding the measure and intrinsic for these cases. For $K = 2$, the intrinsic for the path is $L(\rho) = \rho^2 \log\log\log(1/\rho)$, while for $K \geq 3$, the intrinsic for the path is $L(\rho) = \rho^2 \log\log(1/\rho)$. (The more complicated expression for $K = 2$ is due to the fact that the path is able to cross over itself infinitely often.) He also showed that for $K = 1$, the graph has zero measure with $h(\rho) = \rho^{3/2}$.

In view of the correspondence between Taylor's rigorous definitions of Brownian motion and the concept of the path integral as introduced by Feynman, it is comforting to note that our heuristic analysis yields answers in agreement with his work.

5. Conclusions

We have investigated the fractal nature of the dominant paths contributing to Feynman's path integral for the quantum oscillator in both one and two dimensions. A naive application of Mandelbrot's formula for fractal dimension yields $D = 1.5$ in both cases, in contrast to the result of Abbott and Wise, who arrived at the result $D = 2$ for quantum motion, albeit in a different context. More worrying was the apparent contradiction with the results of Taylor, who predicted $D = 1.5$ and $D = 2$ for graphs of one- and two-dimensional Brownian motion respectively. Since the quantum mechanical case differs only by an irrelevant potential function, we would expect these results to be true for quantum paths. However, a clearer examination of the connection of Mandelbrot's definition of fractal dimension shows that the appropriate resolution must be chosen with care. For the one-dimensional oscillator, the resolution is indeed the time separation, a, and we have $D = 1.5$. For the two-dimensional oscillator, the appropriate resolution is $a^{1/2}$ corresponding to the average step size resulting in $D = 2$, in agreement with that expected from the work of Taylor.

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