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Structure and Uncomputability in One-Dimensional Maps

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Abstract. We study uncomputable behavior for families of diffeomorphisms of the circle and unimodal maps of the interval. We find that the set of parameters that correspond to simple behavior contains a set that is open and dense. The set that corresponds to uncomputable behavior has positive measure.

1. Introduction

In this paper we study the computational properties of families of one-dimensional maps. We find that even simple maps can display an abundance of both uncomputable behavior and computationally simple behavior. Complex behavior, which is computable but not simple, appears to be rare.

We define computability in terms of the natural symbolic dynamics on the map. A map displays simple computational behavior if its symbolic dynamics can be computed on a machine with finite memory. It is uncomputable if there is no Turing machine that can compute its dynamics.

In terms of formal language theory [11] *simple* refers to regular languages that can be computed on a finite automata, and *uncomputable* refers to languages that are not recursively enumerable. *Complex* specifies computable languages that are not regular.

Specifically, for certain families of homeomorphisms of the circle and unimodal maps of the interval we find that the set of parameter values for which the dynamics is simple (Δ_s) contains a set that is open and dense. The set of parameters that generate uncomputable dynamics (Δ_u) has positive measure. The remaining set (Δ_c), which contains those parameter values for which the dynamics is computable but not simple, has zero measure. We believe that these results are generic to families of one-dimensional maps. Analogous structures seem to exist in higher dimensional maps.

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2. Outline of paper

In this paper we analyze the computational complexity of parameterized mappings. It turns out that if we choose a mapping at random from these families then typically one of two things will occur. First, the mapping could be simple. That is, its symbolic dynamics could be simulated on a computer with only a finite amount of memory. This does not imply that the map itself is computable. As is well known, most parameter values are uncomputable, yet many of these lead to simple symbolic dynamics. Nor does it imply that the map is periodic. The logistic map $f(x) = rx(1-x)$ at $r = 4$ is fully chaotic, yet its symbolic dynamics is very simple in the computational sense. Second, the mapping could be uncomputable. This implies that no computer could exactly simulate its symbolic dynamics.

To prove that something is simple we explicitly construct the computer (finite automaton) that computes it. However, proving that a mapping's symbolic dynamics is uncomputable is itself uncomputable in general, thus no explicit proof of its uncomputability can exist. We prove uncomputability by using a counting argument. If we have an uncountable set of distinct symbolic dynamics, then most of these must be uncomputable since there are only a countable number of computer programs.

The proofs in this paper proceed in the following manner. First, we explicitly construct finite automata for a dense open subset of parameter space. Then we show that the remaining set of symbolic dynamics is uncountable, and thus "most" of these are uncomputable. However, showing that "most" actually correspond to positive measure is nontrivial. It is possible to construct families of maps that are arbitrarily close to those considered in this paper, which have zero measure of uncomputability.

3. Definitions

We define the symbolic dynamics of a map in the standard way [4]. Given a space X and a map $T : X \to X$ we define a partition $P = \{P_0, P_1, \ldots, P_{n-1}\}$ where $P_i \subset X$, $P_i \bigcap P_j = \emptyset$, and $\bigcup P_i = X$. We then study the sequence of partition elements visited by an orbit of the map.

In order to study the computational properties of a dynamical system we must first define *words* and *languages* and the operations on these structures. We can then apply the techniques of formal language theory to analyze these languages. (These definitions are based on the work of Crutchfield and Young [6, 7]).

Definition. A *word* is a finite sequence of partition elements: $w = w_1 w_2 \ldots w_n$, $w_i \in P$ $(1 \leq i \leq n)$. The length of a word is denoted by $|w| = n$.

Definition. Given a word $w = w_1 w_2 \ldots w_n$, the *shift operator* is defined in the following way: $\sigma^m w = w_{m+1} w_{m+2} \ldots w_n$, $m < n$.

Definition. The *language* L generated by a map T (with partition) is the set of allowed words for that map. That is,

$$L(T) = \{w \mid \exists x \text{ such that } \forall 0 < i \leq |w|,\ T^i x \in w_i\}$$

We associate the computational properties of this language with the dynamical system (T, X, P). In particular the dynamics of a map is *uncomputable* if the associated language is not recursively enumerable, and the dynamics is *simple* if the language is regular.

We now define the families of maps that we will study.

Definition. A *parameterized map* is set of functions $T = \{T_r\}$ with a smooth projection $\pi : T \to \Delta$, where Δ denotes the *parameter space.*

We will consider the case when Δ is an interval on the real line, with the standard Lebesque measure.

Definition. The *hump map* is a family of C^1 unimodal maps from the interval $[0,1]$ to itself defined by $T_r x = 2^{k-1} r \left((1/2)^{(1+k)} - |x - \frac{1}{2}|^{(1+k)}\right)$, where $0 < k$ and $r \in \Delta = [2,4]$. The partition for the hump map is $\{[0,1/2), 1/2, (1/2,1]\}$, which we will denote by $\{L, C, R\}$, respectively.

The hump map reduces to the *tent map* when $k = 0$, and to the *logistic map* $T_r x = rx(1-x)$ when $k = 1$. Both of these are well-known examples of unimodal maps.

Definition. The circle map is a set of diffeomorphisms from the circle S^1 to itself defined by $T_r x = x + r + (k/2\pi)\sin(2\pi x) \pmod 1$ $[x \in 0, 1)$, where $0 < k < 1$ and $r \in \Delta = [0,1]$. The partition for the circle map is $\{[0, T(0)], (T(0), 1)\}$, which we label $\{L, R\}$, respectively.

Given the above definitions we now formally define our computational classes.

Definition. Given a parameterized map T with a partition, we define the set Δ_s to be the set of parameter values for which the dynamics is *simple*, and can be computed on a finite automaton.

Definition. Given a parameterized map T with a partition, we define the set Δ_c to be the set of parameter values for which the dynamics is *complex*; that is, the language generated by the symbolic dynamics is computable on a machine with infinite memory, but is not simple.

Definition. Given a parameterized map T with a partition, we define the set Δ_u to be the set of parameter values for which the dynamics is *uncomputable*; that is, the language generated by the symbolic dynamics is not recursively enumerable on a Turing machine.

4. Results

In this section we present our main results. We find that for both the logistic map and the circle map both sets Δ_s and Δ_u have positive measure, and therefore both simple and uncomputable dynamics occur for a wide range of parameter values. The simple dynamics typically occur on a set containing an open and dense subset. Formally we have the following theorems.

Theorem C1. *For the circle map with* $0 < k < 1$*:*

(*a*) *Both* Δ_s *and* Δ_u *have positive measure.*

(*b*) *The set* Δ_s *contains a set that is open and dense. The set* Δ_c *has zero measure, and is relatively dense with respect to* Δ_u.

(*c*) *At* $k = 0$, *the set* Δ_u *has full measure, and this decreases monotonically as* k *is increased, for* k *small. At* $k = 1$ *the set* Δ_u *has full measure.*

Theorem H1. *Given the hump map, then*

(*a*) *For* $k > 0$, Δ_s *has positive measure. For* $k = 1$, Δ_u *has positive measure.*

(*b*) *Assume that* $k = 1$ *and* Δ *is restricted to* $[4 - \epsilon, 4]$, ϵ *sufficiently small. Then:*

The set Δ_s *contains a set that is open and dense;*

The set Δ_c *has zero measure. It is relatively dense with respect to* Δ_u.

(*c*) *At* $k = 0$ *the set* Δ_u *has full measure.*

Remark. It is an open question whether there exists a value of k for which Δ_s contains the full measure.
Remark. If we assume that the set of r which generate periodic kneading invariants is dense (as conjectured in [14], page 547) we can remove the restriction on Δ in Theorem H1*b*.

Before proceeding we point out that these ideas are far more general than is shown in the above theorems. The techniques used to prove Theorem H1 for the logistic map apply to any generic parameterized sets of C^1 unimodal maps. This is because the kneading calculus applies to any C^1 unimodal map, and the proofs should carry through in most cases. Even multi-modal maps should obey these theorems since the kneading calculus can be generalized to them without much difficulty. Similarly, these ideas should apply to most parameterized diffeomorphisms of the circle, as long as their parameterization allows a full range of winding numbers to be obtained. For example if $\sin(2\pi x)$ in the circle map is replaced by any $g(x)$ that is nonlinear, periodic, and smooth, then Theorem C1 should hold.

The surprising generality of these results leads us to conjecture that the structures we have described occur for generic smooth families of one-dimensional maps.

5. Proof for the circle map

In this section we prove Theorem C1. First we need some basic definitions.

Definition. Given the partition $\{L, R\} = \{[0, T(0)), [T(0), 1)\}$, we define the winding number to be

$$\rho(T) = \lim_{n\to\infty} \frac{1}{n} N(x,n)$$

where $N(x,n)$ is the cardinality of the set $\{i \mid T^i(x) \in L,\ 0 \le i \le n\}$. This limit exists and is independent of x (see [10]).
Remark. Note that we consider an intrinsic definition of the winding number that does not use a lift of S^1.

The winding number completely determines the language for the circle map. Any two diffeomorphisms with the same irrational winding number are topologically conjugate [2], and as the language generated by a map is a topological invariant they must have the same language. In [8] we explicitly compute the language generated for any irrational winding number. In appendix A we compute it for rational winding numbers. Thus we can easily check that the map between winding numbers and languages is one-to-one.

Lemma C1. *If two diffeomorphism of the circle have different winding numbers (bmod 1/2) then the languages they generate are different. (If two languages have winding numbers that different by exactly 1/2 then they are isomorphic by exchanging R and L.)*

Proof. See appendix B and [8] for the explicit construction of these languages. ∎

The structure of the parameter space has been well studied in other contexts. The following results are crucial to our theorem.

Proposition C1. *The following has been proved by Arnold [2]:*

(a) *For $0 < k$, k small, there exists a dense set of positive measure in Δ, such that the winding number is rational and the complement of this set is a Cantor set that contains all irrational winding numbers.*

(b) *The map from irrational winding numbers to parameter values is one to one.*

(c) *The measure of the set of r that lead to rational winding numbers is zero for $k = 0$ and increases monotonically for k small.*

Proof. See [2]. ∎

Proposition C2. The following was discovered in numerical experiments by Jensen et al. in [13] and later proved by Swiatek under very general conditions in [15]: *For $k = 1$ the measure of the set of r that lead to rational winding numbers contains the full measure.*

Proof. See [15]. ∎

The following proposition describes the set Δ_s.

Proposition C3. *In the circle map rational winding numbers generate regular languages.*

Proof. The language is constructed in appendix A. ∎

The proof of Theorem C1 follows quite easily from the above results.

Proof of Theorem C1.

(*a*) The set Δ_u has positive measure because the union of Δ_u and Δ_c has positive measure by Proposition C1; but Δ_c has zero measure, as will be shown in part (*b*).

(*b*) By Proposition C1, part (*a*), we know that the set of rational winding numbers contains a set that is open and dense. By proposition C3 the same must be true of Δ_s.

There are only a countable number of languages and thus a countable number of irrational winding numbers that are computable. Since the map between irrational winding numbers and parameters is one-to-one this set must have zero measure. Finally, we note that the computable irrationals are dense in the set of irrationals, and maps with computable winding numbers are obviously computable.

(*c*) At $k = 0$ the map from Δ to languages is one-to-one. As the set of computable languages is countable, so is its inverse image. As k is increased the measure of the rational winding numbers increases. At $k = 1$ the set of rational winding numbers has full measure. Applying Proposition C3 we get the desired result. ∎

6. Proof for the hump map

In this section we prove Theorem H1. The proof is similar in structure to that for the circle map. It depends on properties of the kneading invariant K. The kneading invariant is the symbolic trajectory of the critical point of the map, and completely determines the symbolic dynamics of the map. Our explanation of the kneading invariant will be brief as it is well described in several references. (See [14] and [5].) We will follow the notation of Collet and Eckmann [5].

Definition. The *kneading invariant* is the semi-infinite symbol sequence generated by the critical point ($x = 1/2$) of the hump map. (We always consider infinite kneading sequences, namely the extended itinerary in Collet and Eckmann's terminology.)

Definition. We define an *ordering* on the set of sequences in the usual manner. Consider two sequences s and t. Let i be the smallest integer for which $s_i \neq t_i$, if it exists. If i does exist, define $L < C < R$. Now consider the prefix $p = s_1 s_2 \ldots s_{i-1} = t_1 t_2 \ldots t_{i-1}$. If p has an even number of Rs and $s_i < t_i$, or p has an odd number of Rs and $s_i > t_i$, then we say that $s < t$; otherwise $s > t$. Now if i as defined above does not exist, then the shorter string is defined to be the smaller one. If it does not exist and both strings are the same length or infinite then they are equal.

Now we show that the kneading invariant defines the language generated by the hump map.

Proposition H1. *Consider the hump map:*

(*a*) *The kneading invariant completely determines the language generated by the map.*

(*b*) *There is a one-to-one map from kneading invariants to languages.*

Proof.

(*a*) Collet and Eckman [5] show that the set of allowed sequences in a C^1 unimodal map with kneading sequence K is just the set of all sequences s that satisfy the condition $\sigma^m s < K$, $\forall m > 0$. The language is just the set of all words that occur in these sequences, since the hump map is C^1.

(*b*) Assume we have two maps with different kneading sequences $K_1 < K_2$. Assume that K_1 and K_2 disagree on their $(n-1)$th symbol. Now let w be the first n letters of K_2. W is a word in the language generated by K_2 since it is part of K_2. However, it is not in the language generated by K_1 since any extension of it to an infinite sequence creates a sequence that is larger than the kneading invariant. ∎

We show that Δ_s is related to periodic kneading sequences.

Proposition H2. *Periodic kneading invariants give rise to regular languages.*

Proof. We construct the finite automaton that recognizes the language in appendix B. ∎

That the aperiodic kneading sequences have positive measure was first shown by Jacobson [12]. We require a slightly stronger result.

Proposition H3. The following has been proved by Benedicks and Carlson [3]. *For $k = 1$ and Δ restricted to $[4 - \epsilon, 4]$, ϵ sufficiently small, the set of* *periodic kneading invariants contains a set that is open and dense, and the set of aperiodic kneading invariants has positive measure.*

The proof for Theorem H1 readily follows from the above results.

Proof of Theorem H1.

(*a*) For $k > 0$, periodic kneading sequences correspond to maps with stable periodic orbits. It is easy to show that superstable periodic orbits exist for hump maps with $k > 0$. Now for small changes in r the periodic orbit must persist; therefore the periodicity of the kneading sequences of these maps also persists and, by Proposition C3, Δ_s must have positive measure. By Proposition H3 and part (*b*) of this theorem Δ_u must have positive measure.

(*b*) The first part follows trivially from Proposition H2 and Proposition H3.

In [14] it is shown that the map from Δ to kneading invariants is monotonic. Since Δ_s is open and dense, this implies that Δ/Δ_s is completely disconnected. Thus the map from Δ/Δ_s to kneading invariants must be one-to-one. The set of computable languages is countable, so the set Δ_c must be countable. Therefore it has measure zero.

(*c*) At $k = 0$ the map from Δ to kneading invariants (and hence to languages) is one-to-one. Therefore both Δ_s and Δ_c are countable and must have zero measure. ■

7. Some thoughts on higher dimensional dynamics

In this section we discuss the extension of these ideas to two-dimensional maps. The extrapolation of our results for one-dimensional systems to higher dimensions is meant to be suggestive, as perhaps higher dimensional systems have a generic computational structure.

In R^4 consider an integrable hamiltonian H. In this case the phase space is foliated by tori (with occasional degeneracies). If we take a Poincaré section of the space, at constant energy, we get a map from R^2 to itself, which is essentially a collection of independent diffeomorphisms of the circle. Most of these maps will correspond to uncomputable winding numbers with uncomputable dynamics. Thus for typical integrable systems the full measure of the space will contain uncomputable dynamics, with the simple dynamics occurring on a dense set of zero measure.

Now consider a small hamiltonian perturbation that renders the system non-integrable. By the KAM theorem [1] we know that each circle with a rational winding number will break up, usually into stable and unstable periodic orbits. The stable orbits should be dense in the space, and each one should have an open neighborhood of stability. These will correspond to an open dense subset of computable behavior.

However, a finite measure of the irrational tori will still remain. Most of these will be uncomputable.[1] The computable ones should have zero measure. Thus we have the analog of our results for parameter space in an actual system.

8. Conclusions

We have shown that for two well-known and well-studied maps the set of parameter values on which the dynamics is simple is dense and has nonzero measure. Also, uncomputable dynamics also occurs with positive measure. The set of simple dynamics grows as the nonlinearity of the map increases. Thus, nonlinearity actually simplifies the description of the dynamics.

Complex but computable dynamics occurs very rarely in these maps (zero measure). We call these maps critical and believe that they are very important in organizing the structure of parameter space (see [8]).

These ideas seem to extend to higher dimensional systems. It appears that open dense sets of computable behavior could intermix with uncomputable behavior in generic chaotic systems, and both types of behavior have positive measure. This could have strong implications for the study of dynamics.

Finally we emphasize that these results appear to be generic to families of one-dimensional maps. The structure of the computable and uncomputable sets does not seem to depend on the detailed structure of the maps.

Appendix A. Construction of finite automata for rational winding numbers

In this appendix we construct the regular language that is generated by the circle map with a rational winding number. Our construction applies to all diffeomorphisms of the circle with a finite number of periodic orbits. The circle map has a single stable periodic orbit and a single unstable one.

The result would be trivial if the point $x = 0$ was part of a periodic orbit. Then the symbolic dynamics would just be a repeating sequence of Rs and Ls. However, this is unusual and typically the language is more complicated.

First we will need some elementary results from the theory of dynamical languages [9].

Definition. Given a word $w = w_1 w_2 \ldots w_n$, a word v is a *substring* of w if $v = w_i w_{i+1} \ldots w_j$ for $0 < i < j \leq n$.

Definition. Given a language L, the *substring closure* of L is the language $SC(L) = \{v \mid \exists w \in L$ such that v is a substring of $w\}$.

Proposition A1. *Assume that the language L is regular. Then $SC(L)$ is regular.*

[1]Note that the uncomputable tori *do not* correspond to the *most irrational* tori discussed so much in reviews of KAM theory. For example, the golden mean torus is easily computable on a Turing machine.

Proof. See [9]. ∎

We now define a set of useful intervals that will allow us to deduce the language.

Definition. Consider a diffeomorphism of the circle with a finite number of periodic orbits with rational winding number $\rho = p/q$, where $p, q \in Z^+$ and p/q in lowest terms. Define its *fundamental stable* (respectively *unstable*) *orbit* to be the set $O_s = \{x_1, x_2, \ldots, x_q\}$ (resp. $O_u = \{y_1, y_2, \ldots, y_q\}$) where O_s (resp. O_u) is the stable (resp. unstable) periodic orbit such there are no points from O_s and O_u between x_1 (resp. y_1) and 0.

Definition. The *fundamental intervals* $I_1, I_2, \ldots, I_q$, are defined in the following way. If $x_1 < 0$ then $I_1 = (x_1, x_2)$, otherwise $I_1 = (x_q, x_1)$. Now define $I_{i+1} = T(I_i)$ for $0 < i < q$.
Remark. Note that $T(I_q) = I_1$, and that the union of the fundamental intervals and the fundamental orbit is S^1.

Now we study the dynamics on the interval to get the symbolic dynamics.

Proof of Proposition C3. First notice that, for the intervals $I_3, \ldots, I_q$, each interval is contained in a single element of the partition. Thus define A to be this sequence of partition elements:

$$A = P_{i_1} P_{i_2} \ldots P_{i_{q-2}} \text{ such that } I_j \subset P_{i_j}.$$

We consider this as a sequence of Rs and Ls.

In the remainder of the proof we will assume that $x_1 > 0$ for notational convenience. The case $x_1 < 0$ can be treated similarly; the case $x_1 = 0$ is trivial. Divide the intervals into three sections:

$$I_{1a} = (x_q, y_1],\ I_{1b} = (y_1, 0],\ I_{1c} = (0, x_1)$$
$$I_{2a} = (x_j, y_2],\ I_{2b} = (y_2, T(0)],\ I_{2c} = (T(0), x_1)$$

where x_j the nearest element of the fundamental stable orbit to the left of y_2. These divisions have been chosen such that each one corresponds to a single element of the partition:

$$I_{1a} \in R,\ I_{1b} \in R,\ I_{1c} \in L$$
$$I_{2a} \in L,\ I_{2b} \in L,\ I_{2c} \in R$$

Now we can write down the symbolic sequences that are possible.

Starting from I_1 we get the following regular expressions for the sequences, starting from different parts of I_1:

$$I_{1a} : (RLA)^*, \qquad I_{1b} : (RLA)^*(LRA)^*, \qquad I_{1c} : (LRA)^*,$$

and the regular expressions for the fundamental orbits are:

$$O_s = (RLA)^*, \qquad O_u = (LRA)^*.$$

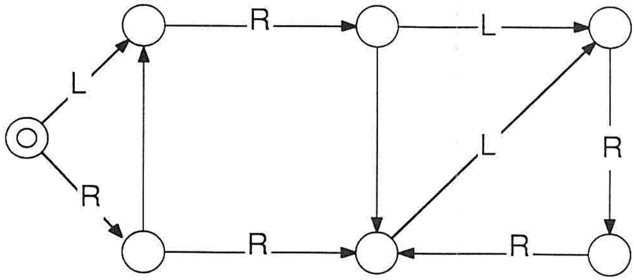

Figure 1: The minimal DFA that accepts the language generated by the circle map for $p = 1/3$.

Clearly the union of these regular expressions is $L_1 = (RLA)^*(LRA)^*$. The full language is just the set of all substrings in this union, $SC(L_1)$. This language is regular by Proposition A1. ■

In figure 1 we show the minimal deterministic finite automaton that accepts the language for $\rho = 1/3$.

Appendix B. Construction of finite automata for periodic kneading sequences

In this appendix we construct the automata that accepts the language for any periodic kneading invariant.

Definition. Given a kneading invariant K the *language L generated by K* is the set of words w such that, if $|w| = n$, then $\sigma^m w < K$, $\forall m \leq n$.

Definition. A finite automaton $A = (Q, q_0, F, \Sigma, \delta)$ is a set Q of states with an initial state $q_0 \in Q$ and final states $F \subset Q$, Σ is an alphabet, and δ is the transition function from $Q \times Q \to 2^{\Sigma}$. A word w is in the language generated by A if there exists a sequence of states $q_{i_1} q_{i_2} \ldots q_{i_n}$ such that $w_i \in \delta(q_{j_i}, q_{j_{i+1}})$ for $i \leq n$, $q_{i_0} = q_0$, and $q_{i_n} \in F$.

We now construct an automaton that accepts the language.

Lemma B1. *Given a periodic kneading invariant with period p, there is a deterministic finite automaton (DFA) that accepts the language L_0, which is the set of all words that satisfy $\sigma^{jp} w < K$, $j \in Z$, $|w| = n$, $0 \leq jp < n$.*

Proof. We construct the DFA explicitly. The DFA has $3p - 1$ states, which we divide into 3 classes: kneading, critical, and small. The basic idea is that the kneading states track whether or not the word is larger than the kneading invariant. If a letter is seen that makes the word less than K, we then switch over to the small states, which allow any letter. If we see a C we move to the critical states, which check whether the following sequence is exactly the kneading sequence. (The sequence following a C must be the kneading sequence.) Finally (except in the case a C is seen) the DFA is strictly periodic of period p, and thus repeats every p symbols. Thus after every p symbols we start the process again, which is equivalent to checking all p-shifts of the word.

The set of states is

$$Q = k_1, k_2, \ldots, k_p, c_1, c_2, \ldots, c_p, s_2, s_3, \ldots, s_p$$

where k_1 is the initial state and all states are final. The transitions are defined so that the kneading and critical states accept the kneading sequence:

$$\delta(k_i, k_{i+1}) = \{K_i\}, i < p,$$
$$\delta(c_i, c_{i+1}) = \{K_i\}, i \leq p.$$

Now we have two possibilities depending on whether $K_p = C$. If $K_p = C$ then

$$\delta(k_p, k_1) = \left\{ \begin{array}{l} \{L\},\ L < C \\ \{R\},\ R < C \end{array} \right\} \quad \text{and} \quad \delta(k_p, c_1) = \{C\}$$

where $L < C$ if the number of Rs in the first $p-1$ symbols of the kneading sequence is even; otherwise $R < C$. If $K_p \neq C$ then

$$\delta(k_p, k_1) = \{K_p\}.$$

Now we define the transitions for the small states,

$$\delta(s_i, s_{i+1}) = \{R, L\} \qquad 1 < i < p,$$

and from the small states to the other states,

$$\delta(s_p, k_1) = \{R, L\}, \qquad \delta(s_i, c_1) = C \qquad 1 < i < p.$$

Finally we define the transitions from the kneading states to the other states,

$$\delta(k_i, s_{i+1}) = \left\{ \begin{array}{l} \{L\},\ L < K_i \\ \{R\},\ R < K_i \end{array} \right\}$$
$$\delta(k_i, c_1) = \{C\} \quad \text{if} \quad C < K_i.$$

This is the complete set of states and transitions, which have been constructed to accept L_1. ∎

Lemma B2. *Given a periodic kneading invariant with period p, there is a DFA that accepts the language L_m. L_m is the set of all words that satisfy $\sigma^{jp+m}w < K$, $j \in Z$, $jp + m < |w|$.*

Proof. For $m > 1$ this DFA is a slight modification of the one in the previous lemma. All we do is add m new states at the beginning of the DFA, which discard the first m letters. Formally, we add the following *delay* states to

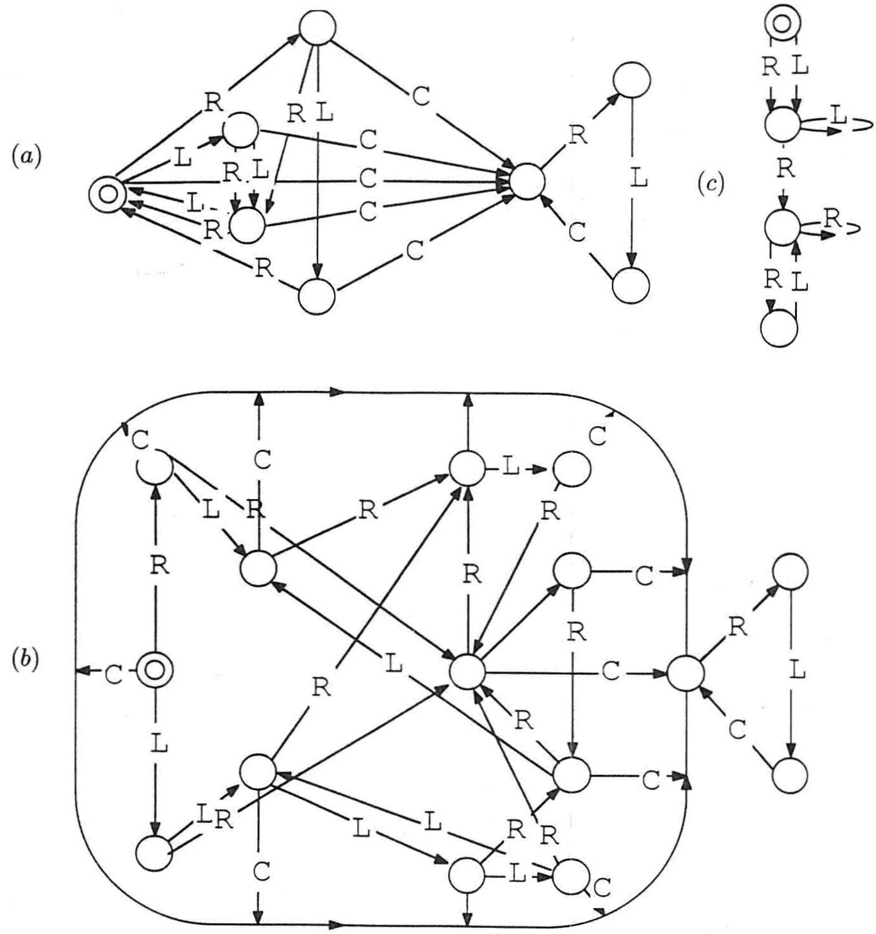

Figure 2: For the logistic map at $r = 3.83\ldots$, which corresponds to the hump map when $K = (RLC)^*$: (a) The DFA for the language L_0 constructed in Lemma B1. (b) The DFA constructed for the full language L. (c) The minimal DFA for the full language, which accepts the same language as (b).

the previous DFA, $d_1, \ldots, d_m$, with the modification that d_1 is now the new initial state. The new transitions are

$$\delta(d_i, d_{i+1}) = \{R, L, C\} \quad i < m, \quad \text{and} \quad \delta(d_m, k_1) = \{R, L, C\}. \quad ■$$

Proposition B1. *The intersection of a finite number of regular languages is regular.*

Proof. This is proved in [11]. ■

Remark. There is a simple algorithm for constructing a DFA that accepts the intersection of a finite number of languages generated by known DFAs [11].

Applying the above proposition to the languages defined in Lemmas A1 and A2 we get the desired result.

Proof of Proposition H2. The language is just the intersection of the previously defined languages,

$$L = \bigcap_{i=0}^{p-1} L_i.$$

Because finite intersections of regular languages are regular, the full language must be regular and hence has a description in terms of a DFA. ■

In figure 2 we show the DFAs for L_1 and L for logistic map at $r = 3.83\ldots$, which corresponds to the supercritical period-three orbit, $K = RLCRLC\ldots$.

Acknowledgments

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