Cellular Automata and Continuous Functions: Negative Results

Yuri Ozhigov[*]

Department of Mathematics,
Moscow State Technological University "Stankin,"
Vadkovsky per. 3a, 101472, Moscow, Russia

Abstract. Let $\omega = \{0, 1, \ldots, n-1\}$ be a finite alphabet, $D_N = \{1, 2, \ldots, N\}$, and $B_N = \{x \in [0, 1] \mid \exists k \in \mathbb{N} : x = k/n^N\}$.

A configuration is a function of the form: $\xi : D_N \to \omega$, and C_N is the set of all configurations. Two configurations ξ_1 and ξ_2 are near if $d(\xi_1, \xi_2) = (N - A)/N$ is small, where $A = \sup\{p \mid \exists i \in \{0, 1, \ldots, N\} : \forall k = i + 1, i + 2, \ldots, i + p \le N \ \xi_1(k) = \xi_2(k)\}$.

The following results are proved.

1. There is no sequence of functions $\phi_N : C_N \to B_N$ such that ϕ_N and ϕ_N^{-1} uniformly converge to continuous functions in such a topology.

2. Evolutions of cellular automata (CA) cannot be approximated by the superpositions of real continuous functions.

In the proofs of these results advantage was taken of some CA acting in $\mathbb{Z}$ and in D_N with a stationary boundary condition.

1. Introduction and definitions

Cellular automata (CA) are general discrete models of natural processes such as the process of heat transfer or spreading of waves, and simultaneously CA are simple models of computations. CA were introduced by S. Ulam in [7] and J. von Neumann in [6]. Since then different aspects of CA have been investigated by many scientists (e.g., [8]).

One-dimensional CA of radius 1 is defined by the following.

1. Space $\mathbb{Z}$, having elements which are called cells.

2. Finite set $\omega_1 = \{a_0\} \cup \omega$, of possible states of any cell, where $\omega = \{0, 1, \ldots, n-1\}$, $n > 1$.

3. Function $\mathcal{A} : \omega_1^3 \to \omega_1$, where $\mathcal{A}(x, y, z) = a_0$ iff $y = a_0$.

---

[*]Electronic mail address: y@oz.msk.ru.

This CA determines the evolution of space so that the states of each cell i evolve in time steps $t = 0, 1, \ldots$ according to the states of its neighborhood. This dependence is determined by $\mathcal{A}$ provided the neighborhood of i consists of three nearest cells: $i - 1$, i, and $i + 1$.

Note that one-dimensional CA with radius $R \geq 1$ can be defined along similar lines, only a function $\mathcal{A}$ will have the form $\mathcal{A} : \omega_1^{2R+1} \to \omega_1$ and the neighborhood of i will consist of the cells: $i - R, i - R + 1, \ldots, i, i + 1, \ldots, i + R$.

We shall treat evolutions of $\mathcal{A}$ in limited domains of space of the form: $D_N = \{1, 2, \ldots, N\}$, $n = 1, 2, \ldots, \infty$ with a stationary boundary condition. Such an evolution is a function of the form: $\xi : D_N \times \mathbb{N} \to \omega$, where $\forall t = 0, 1, \ldots, \ i = 2, \ldots, N - 1$

$$\xi(t + 1, i) = \mathcal{A}(\xi(t, i - 1), \xi(t, i), \xi(t, i + 1)), \tag{1}$$

$$\xi(t + 1, 1) = \mathcal{A}(a_0, \xi(t, 1), \xi(t, 2)), \tag{2}$$

$$\xi(t + 1, N) = \mathcal{A}(\xi(t, N - 1), \xi(t, N), a_0). \tag{3}$$

Evolution in $\mathbb{N} = D_\infty$ is defined by equations (1) and (2), evolution in $\mathbb{Z}$ is defined by equation (1). The set of all possible configurations $\xi : D_N \to \omega$ is denoted by C_N. The configuration in moment t of evolution $\xi(t, i)$ is $\xi_t = \xi(t, i)$. If $b \in C_N$, $b = \xi_0$, we designate ξ_1 by $(b)'$, or by $(b)'_N$.

S. Wolfram in 1984 asked the question: What is the relation between CA and continuous systems? (see [8]). In particular, [2] and [5] deal with treating CA as continuous functions, their applications to dynamical systems are treated in [3]. One of the possible relations between these two types of mathematical objects was demonstrated in [4], where configurations were represented by polynomials $A^{(t)}(x) = \sum_i \xi(t, i) x^i$ and CA action was represented by multiplication of $A^{(t)}(x)$ by the fixed polynomial $T(x)$, which depends on the CA.

This paper shows the impossibility of having a simpler way with configurations represented by real numbers and each step of evolution represented by the action of a continuous function.

2. Evolutions in unlimited domains

In this section we consider CA in unlimited domains $\mathbb{N} = C_\infty$ or $\mathbb{Z}$. Any possible configuration $b \in C_\infty$ may be denoted as the sequence

$$b(1), b(2), \ldots$$

of natural numbers from ω.

At first glance, the natural correspondence between possible configurations $b \in C_\infty$ and real numbers $x : 0 \leq x \leq 1$ can be established if any configuration b is associated with a real number x_b which has arithmetic notation $0.b(1)b(2)\ldots$ in an n-based number system with the figures $0, 1, \ldots, n - 1$.

The function $x : C_\infty \to [0, 1]$ defined by

$$x(b) = 0.b(1)b(2)\ldots$$

is a direct arithmetic notation of configurations.

The idea, in outline, is to associate with CA $\mathcal{A}$ the special real function $f_{\mathcal{A}}(x)$ of the form $[0,1] \to [0,1]$ defined by the following

$$f_{\mathcal{A}}(x(b)) = x((b)'_{\infty}). \tag{4}$$

This approach has the grave drawback that there are different configurations of the form

$$\bar{b} = (t_1, t_2, \ldots, t_k, n-1, n-1, \ldots), \ t_k < n-1 \text{ and}$$
$$\bar{c} = (t_1, t_2, \ldots, t_k+1, 0, 0, \ldots),$$

where $x(\bar{b}) = x(\bar{c})$ and so the natural definition by equation (4) is incorrect for the majority of cases. Moreover, this approach can give only discontinuous functions.

A function $\mathcal{A}$ from the definition of CA induces the mapping on the set of all configurations which will be denoted by the same letter $\mathcal{A}$.

We say that a function $f_{\mathcal{A}} : B \to B$, where $B = \mathbb{R}$ or $B = [0,1]$ is a real presentation (RP) of a CA $\mathcal{A}$ if there exists one-to-one correspondence $\phi : B \to C_{\infty}$ such that the following diagram is closed:

$$
\begin{array}{ccc}
C_{\infty} & \xrightarrow{\ \mathcal{A}\ } & C_{\infty} \\
\phi \uparrow & & \uparrow \phi \\
B & \xrightarrow[f_{\mathcal{A}}]{} & B
\end{array}
$$

Theorem 1. *There exists CA $\mathcal{A}$ with no continuous RP.*

Proof. For any function F let F^s denote its iteration: $F^0(x) = x$, $F^{s+1} = F(F^s(x))$. For any RP $f_{\mathcal{A}}$ and $s \in \mathbb{N}$ we have:

$$\forall \xi \in C_{\infty} \quad \mathcal{A}^s(\xi) = \phi(f_{\mathcal{A}}^s(\phi^{-1}(\xi))).$$

Let $\mathcal{A}(i, j, k) \equiv j + 1 \pmod{n}$. Then $\mathcal{A} : C_{\infty} \to C_{\infty}$ has no fixed point and $\mathcal{A}^n$ is identical. Thus, if $f_{\mathcal{A}}$ is continuous, it must be $\forall x \in B \ f_{\mathcal{A}}(x) < x$ or $\forall x \in B \ f_{\mathcal{A}}(x) > x$. Then $f_{\mathcal{A}}^r$ cannot be identical for any $r = 1, 2, \ldots$, which is a contradiction. Theorem 1 is proved. ∎

The case of cyclicity of $\mathcal{A}$ can be excluded if we extend the notion of RP. Let $s \in \mathbb{N}$. A real function $F_{\mathcal{A}} : B \to B$ is called a real s-presentation (RS-P) of CA $\mathcal{A}$, if the diagram

$$
\begin{array}{ccc}
C_{\infty} & \xrightarrow{\ \mathcal{A}^s\ } & C_{\infty} \\
\phi \uparrow & & \uparrow \phi \\
B & \xrightarrow[F_{\mathcal{A}}]{} & B
\end{array}
$$

is closed for some one-to-one mapping ϕ.

Theorem 2. *There exists CA $\mathcal{A}$ such that for any $s \in \mathbb{N}$ $\mathcal{A}$ lacks continuous RS-P.*

Proof. First, consider the case $D = \mathbb{Z}$. Let $\mathcal{A}$ be left shift: $\mathcal{A}(i, j, k) = k$. Suppose that $\mathcal{A}$ has RS-P $F_{\mathcal{A}}$. Then $F_{\mathcal{A}}$ is a one-to-one mapping like $\mathcal{A}$ and thus $F_{\mathcal{A}}$ is monotone. Therefore for any $x \in B$ either $F_{\mathcal{A}}^2(x) = x$ or $\forall l = 1, 2, \ldots$ $F_{\mathcal{A}}^l(x) \neq x$ and the same possibilities take place for $\mathcal{A}^s$ playing the role of $F_{\mathcal{A}}$. But it is impossible because for a configuration

$$a(i) = \begin{cases} 1, & \text{if } i \equiv 0 (\text{mod } p), \\ 0, & \text{if } i \not\equiv 0 (\text{mod } p) \end{cases}$$

and $p = 2s + 1$ $\mathcal{A}^{2s}(a) \neq a$ but $\mathcal{A}^{2s+1}(a) = a$. Case $D = \mathbb{Z}$ is considered.

Now let $D = \mathbb{N}$. The alphabet ω for $\mathcal{A}$ consists of all pairs $\binom{a}{b}$, $a, b \in \{0, 1\}$. We define the CA $\mathcal{A}$ by the following equalities: $\mathcal{A}(\binom{a}{b}, \binom{c}{d}, \binom{e}{f}) = \binom{a}{f}$, $\mathcal{A}(a_0, \binom{a}{b}, \binom{c}{d}) = \binom{b}{d}$. Again, $\mathcal{A}$ induces one-to-one mapping on the set of all configurations, $F_{\mathcal{A}}$ is monotone, and we have the same two possibilities for $\mathcal{A}$ and $F_{\mathcal{A}}$. Define $a(i)$ by the following equations: $a(i) = \binom{1}{1}$, if $i \equiv 0 \pmod{p}$, $a(i) = \binom{0}{0}$, if $i \not\equiv 0 \pmod{p}$. Then we have $\mathcal{A}^{2s}(a) \neq a$, $\mathcal{A}^{2s+1}(a) = a$ as above. Theorem 2 is proved. ∎

Stronger assertions about RPs can be obtained if we set some limits on a function ϕ and on a domain of space for the CA. The case of unlimited domain $D = \mathbb{N}$ is quite simple.

Let us introduce a topology based on the system of vicinities of the form

$$U(q, i_1, i_2, \ldots, i_p) = \{\xi \in C \mid \forall j = 1, 2, \ldots, p \; \xi(q + j) = i_j\}.$$

Then the convergence $\xi_p \to \xi$ $(p \to \infty)$ in this topology implies that $\forall N \; \exists P \; \forall p \geq P \; \forall i = 0, 1, \ldots, N$

$$\xi_p(i) = \xi(i). \tag{5}$$

Any one-to-one mapping $\phi : C_\infty \to [0, 1]$ must have an infinitely large number of points of discontinuity in this topology, because otherwise equation (5) is violated.

3.   The case of limited domains

Let $B_N = \{x \in [0, 1] \mid \exists k \in \mathbb{N} \; x = k/n^N\}$ and let ϕ_N be a one-to-one mapping of the form

$$\phi_N : \; C_N \to B_N. \tag{6}$$

The main question: Is there any sequence of one-to-one mappings of the form in equation (6) such that both sequences ϕ_N and ϕ_N^{-1} converge uniformly if $N \to \infty$? The notion of uniform convergence must be defined more exactly. If $\xi_1 \in C_{N_1}, \xi_2 \in C_{N_2}, N = \min\{N_1, N_2\}$, we put $d(\xi_1, \xi_2) = (N - A)/N$, where $A = \sup\{p \mid \exists i \in \{0, 1, \ldots, N\} : \; \forall k = i+1, i+2, \ldots, i+p \leq N \; \xi_1(k) = \xi_2(k)\}$. If $N_1 = N_2$, then d is a metric on the set C_N.

Let the sequence of mapping in equation (6) have the following properties.

P1. $\forall \epsilon > 0 \ \exists \delta > 0, N_0 \in \mathbb{N} \ \forall N_1, N_2 > N_0, \ \xi_1 \in C_{N_1}, \xi_2 \in C_{N_2}$ if $d(\xi_1, \xi_2) < \delta$, then $|\phi_{N_1}(\xi_1) - \phi_{N_2}(\xi_2)| < \epsilon$.

P2. $\forall \epsilon > 0 \ \exists \delta > 0, N_0 \in \mathbb{N} \ \forall N_1, N_2 > N_0, \ x_1 \in B_{N_1}, x_2 \in B_{N_2}$ if $|x_1 - x_2| < \epsilon$ then $d(\phi_{N_1}^{-1}(x_1), \phi_{N_2}^{-1}(x_2)) < \delta$.

Then we say that the sequence of mapping in equation (6) is a real finite approximation (RFA) of configurations.

RFA, if it exists, would establish a correspondence between real numbers and words in a finite alphabet with natural topology. However, we shall see that such a correspondence is impossible.

Theorem 3. *RFA does not exist.*

Proof. Let equation (6) be RFA, $\mathcal{A}$ be CA, and $f_{\mathcal{A},N}$ be a function which makes the following diagram closed:

$$
\begin{array}{ccc}
C_N & \xrightarrow{\ \mathcal{A}\ } & C_N \\
\phi_N \downarrow & & \downarrow \phi_N \\
B_N & \xrightarrow[f_{\mathcal{A},N}]{} & B_N
\end{array}
\tag{7}
$$

Then P1 and P2 imply that the sequence $f_{\mathcal{A},N}$ converges uniformly to the continuous function $f_{\mathcal{A}}$ on $[0,1]$ if $N \to \infty$.

A function $\mathcal{A}$ is called s-reversible if there exists $N_0 \in \mathbb{N}$ such that CA $\mathcal{A}$ is reversible on all sets C_N, $N > N_0$.

Let $\mathcal{A}$ be s-reversible and $N > N_0$. Then the function $\mathcal{A}$ from equation (7) is reversible and the functions $f_{\mathcal{A},N}$ have reverse functions $f_{\mathcal{A},N}^{-1}$ which converge uniformly to the continuous function $f_{\mathcal{A}}^{-1}$, when $N \to \infty$. Therefore, $f_{\mathcal{A}}$ is monotone. We denote $f_{\mathcal{A}}$ by f.

Lemma 1. *If CA $\mathcal{A}$ is s-reversible then there are the following three possibilities.*

1. $f(x) = x$ for any $x \in [0,1]$.
2. $f(x) = f^{-1}(x)$ for any $x \in [0,1]$.
3. $\exists N \in \mathbb{N}, x_0 \in B_N, \epsilon > 0 \ \forall p = 1, 2, \ldots, \forall M > N \ |f_{\mathcal{A},M}^p(x_0) - x_0| > \epsilon$.

Proof. We shall prove that when f is nondecreasing there are two possibilities: 1 or 3, and if f is decreasing then either 2 or 3.

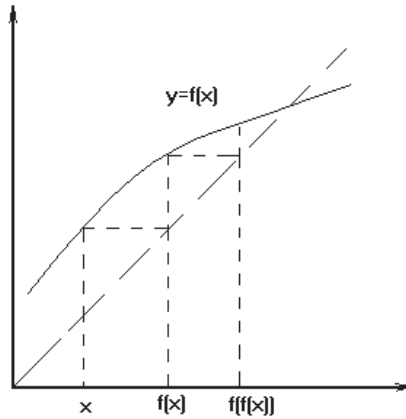**Case 1.** f is a nondecreasing function (Figure 1).
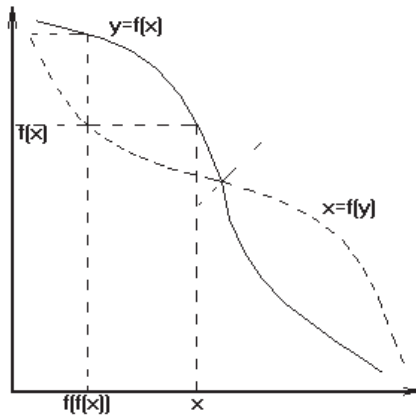
Figure 1: Nondecreasing f.

Figure 2: Decreasing f.

If possibility 1 does not occur, it is sufficient to take x_0 such that $|f(x_0) - x_0| > 0$, $\epsilon = |f(x_0) - x_0|/2$ and find suitable N using P1 and P2.

Case 2. f is a decreasing function (Figure 2).

If possibility 2 does not occur, it is suffucient to take x_0 such that $|f(x_0) - f^{-1}(x_0)| > 0$ and put $\epsilon = \max \{|f(x_0) - f^{-1}(x_0)|, |f^2(x_0) - x_0)|, |f^{-2}(x_0) - x_0)|\}/2$, where $f^{-2}(x) = f^{-1}(f^{-1}(x))$ and after that find appropriate N using P1 and P2. Lemma 1 is proved. ∎

Now let us define CA $\mathcal{B}$.

   The new alphabet $\omega' \cup \{a_0\}$ is introduced by $\omega' = \omega \times \omega$. Elements of ω' are denoted by $\binom{a_i}{a_j}$, $a_i, a_j \in \omega$.

   Rules for CA $\mathcal{B}$ have the following forms:

$$\mathcal{B}\left(\binom{a}{b}, \binom{c}{d}, \binom{e}{g}\right) = \binom{e}{b},$$

$$\mathcal{B}\left(a_0, \binom{c}{d}, \binom{e}{g}\right) = \binom{e}{c},$$

$$\mathcal{B}\left(\binom{a}{b}, \binom{c}{d}, a_0\right) = \binom{d}{b}.$$

Lemma 2. *This automaton $\mathcal{B}$ is s-reversible and the function $f = f_{\mathcal{B}}$ is such that $\exists x \in [0,1]:\ f(x) \neq x,\ f(x) \neq f^{-1}(x)$.*

Proof. Let $\xi_N \in C_N$ be a configuration such that $\forall i = 1, 2, \ldots, N\ \xi_N(i) = \binom{s_i}{x_i}$, where the word $\bar{s}_N = s_1 s_2 \ldots s_N$ does not contain any occurrence of the nonempty word of the form A^l, $l > 2$. This word $\bar{s}_N$ exists for any $N \in \mathbb{N}$ (e.g., [1]). We have $\exists\ N_1\ \forall N > N_1\ d(\xi_N, \mathcal{B}(\xi_N)), d(\xi_N, \mathcal{B}^2(\xi_N)) > 1/2$. Consequently, in view of P1 and P2, if x is a limit point of the set $\{\phi_N(\xi_N) \mid N = 1, 2, \ldots\}$, then $f(x) \neq x,\ f(x) \neq f^{-1}(x)$.
   Lemma 2 is proved. ∎

   Now it is sufficient to note that for any $N \in \mathbb{N}\ \forall x \in B_N\ f_{\mathcal{B},N}^N(x) = x$ and we obtain a contradiction with Lemma 1. Theorem 3 is proved. ∎

References

[1] S. E. Arshon "The Proof of Existence of n-valued Infinite Asymmetric Sequences," *Matematical Sbornic* (Russian), **2**(44)N 4 (1937) 769–779.

[2] R. Feldberg, C. Knudsen, and S. Rasmussen "Recursive Definition of Global Cellular Automata Mappings," *Physical Review E*, **49**, (1994) 1699–1711.

[3] P. Kurka "Dynamical Systems and Factors of Finite Automata," lanl e-print, chao – dyn / 9303006 (1993).

[4] O. Martin, A. M. Odlyzko, and S. Wolfram "Algebraic Properties of Cellular Automata," *Communications in Mathematical Physics*, **93** (1984) 51–89.

[5] C. Moore, "Smooth One-dimensional Maps of the Interval and the Real Line Capable of Universal Computation," *Santa Fe Institute Working Paper* 93-01-001 (1993).

[6] J. von Neumann, *Theory of Self-Reproducing Automata*, edited by A. Burks (University of Illinois Press, 1966).

[7] S. Ulam, "Random Processes and Transformations," *Proceedings of the International Congress on Mathematics*, **2** (1952) 264–275.

[8] S. Wolfram, *Cellular Automata and Complexity: Collected Papers* (Addison Wesley, 1994).