A Small-world Network Where All Nodes Have the Same Connectivity, with Application to the Dynamics of Boolean Interacting Automata

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This paper introduces the equal number of links (ENL) algorithm to generate small-world networks starting from a regular lattice, by randomly rewiring some connections. The approach is similar to the well-known Watts–Strogatz (WS) model, but the present method is different as it leaves the number of connections k of each node unchanged, while the WS algorithm gives rise to a Poisson distribution of connectivities. Motivation for the ENL algorithm stems from interest in studying the dynamics of interacting oscillators or automata (associated to the nodes of the network). Indeed, leaving k unaltered allows one to study how the dynamics of these networks are affected by rewiring only (which gives rise to small-world properties) disentangling its effects from those related to modifying the connectivity of some nodes. The ENL algorithm is compared with that of Watts and Strogatz, by studying the topological properties of the network as a function of the number of rewirings. The effects on the dynamics are tested in the case of the majority rule, and it is shown that key dynamical properties (i.e., number of attractors, size of basins of attraction, transient duration) are modified by rewiring. The quantitative differences between the dynamics of an ENL network and a WS network are discussed in detail. Comparisons with scale-free networks of the Barabasi–Albert type and with completely random networks are also given.

1. Introduction

It is interesting to study the influence of the interaction topology upon the dynamical properties of networks of oscillators or automata, where a variable $x_i(t)$ (which can take either continuous or discrete values)

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is associated to every node i of the network at time t, and there is a deterministic law (e.g., a differential or finite difference equation) which determines the time behavior of x_i. If this equation contains at least one term that depends upon x_j then there is a link from node j to node i.

The dynamical properties of such systems of interacting variables can be influenced by the topological properties of the interaction network. Two well-studied limiting cases are: regular lattices, where only neighboring nodes are linked; and fully random networks, where any two nodes are linked with a given probability [1].

Small-world networks are characterized by path lengths that are short with respect to those of a regular lattice. At the same time they have large clustering coefficients with respect to those of random graphs. Small-world networks represent an interesting class of networks whose properties differ from both limiting cases. Two major "families" of networks of this kind are: (*i*) the Watts–Strogatz (WS) model, where a regular lattice is perturbed by progressively rewiring some connections at random [2, 3]; and (*ii*) the Barabasi–Albert (BA), or scale-free network, which is built by adding new nodes with "preferential attachment," that favors establishing connections with those nodes which are already highly connected [4]. In both cases it is assumed that the connectivity matrix is symmetric (i.e., that links are undirected).

Let k_i be the connectivity, that is, the number of connections, of node i in a network. In the WS and BA cases the resulting network has a distribution of connectivities, which is exponential in the WS model and follows a power law in the BA model [1]. It has been shown that the connectivity of some real-world networks, which display the small-world property, may be better described by a scale-free model (with exponential cutoff), while others follow an exponential distribution [5].

While many studies have been devoted to how networks evolve in time, fewer papers have dealt with the influence of the topology upon the dynamical properties (for a review, see [6]). Interesting recent findings show that the network topology affects the synchronization of nonlinear oscillators [7, 8] and the dynamical properties of random boolean networks [9–10].

It is particularly interesting to look for generic properties associated to changes in the interaction graph. However, whenever one studies the influence of the topology upon the dynamics by either the WS or the BA model, one is actually looking at the combined effect of the following two different mechanisms.

(a) The small-world property (i.e., short characteristic path lengths, high clustering coefficients).

(b) The presence of different connectivity values. Connectivity deeply affects the dynamics in several models, including the majority rule for cellular automata (CA) discussed in section 4, random boolean networks [12], and others.

If one compares the dynamical properties of a network of automata that interact according to the WS or BA topology, with those which can be observed on a regular lattice (where each node has exactly the same number of connections) it can be seen that the effects of the two phenomena overlap.

One may wonder whether properties (a) and (b) always come together, that is, whether a small-world network must necessarily have nodes with different connectivities. We show here that this is not the case, by presenting a network that presents property (a) and not property (b). This allows one to disentangle the contribution of these two aspects from the dynamical properties.

The ENL algorithm generates a small-world network by perturbing a regular lattice, while leaving the number of connections per node unaltered. It has a WS flavor, as some connections are rewired at random, but it differs from the original and gives rise to a probability distribution of the number of connections per node that is a delta function, instead of an exponential function. As in the WS case, the degree of randomness can be tuned by modifying the fraction f of links that are rewired.

The ENL method is presented in section 2 for the case of an undirected graph. It is also shown that the network is indeed of the small-world type, that is, it has a range of values for the parameter f where the characteristic path lengths are short while the clustering coefficient is high.

In section 3 the topological properties of the ENL network are compared with those of the WS model. The procedure for generating ENL networks requires the coupled rewiring of two links at the same time, so some correlations are introduced among the connections. Therefore, by comparing WS and ENL at the same fraction f of rewired links, one finds that the former has a larger degree of randomness. This is quantitatively analyzed in section 3.

Next we come to the issue of the influence of topological changes on the dynamics. We consider the regular lattice ($f = 0$) with a given number of connections k as a reference case, and study the modifications of the attractors and of their basins of attraction as f is increased, while leaving k unaltered.

This problem cannot be studied here in its generality, as it is a huge one. In this paper we report results that have been obtained by studying time-discrete boolean automata. In the regular lattice, these are boolean CA [13–15] (for a recent review see [16]). Although some applications require more than two states (for a recent example see [17]), while other applications require that the internal state of the CA is continuous [18], or is the cartesian product of either discrete or continuous subspaces

[19, 20]. We limit ourselves here to the boolean case, which has proved to be extremely interesting from the theoretical viewpoint [12, 21].

Among the different CA rules, we concentrated upon the so-called "majority rule," which states that an automaton's new state (at time $t + 1$) equals that of the majority of its neighbors at the previous time step. This rule resembles that of a spin that aligns to the magnetic field generated by its neighbors, but the model here is deterministic.

The majority rule has been widely studied in the CA literature [15]. It has also raised the interest of researchers in genetic algorithms, who have been looking for modifications of the rule in order to solve the so-called "majority problem" [22, 23].

The rule is precisely described in section 4, which contains an extensive comparison of the dynamical properties on a regular lattice with those of an ENL network. It is shown that modifying the topological randomness (i.e., incresing the fraction f of rewired links) has a profound influence on the dynamical properties.

In section 5 the modifications of the dynamics that are obtained by randomly rewiring the nodes with the WS model are described and compared to those of the ENL network, and their differences quantified. After taking into account the different randomness of WS and ENL networks at equal f values, it is found that some significant differences still persist. It is tentatively guessed that these differences are due to the distribution of connectivities which is found in WS models. In this section an analysis of the dynamical properties on a scale-free network and on a fully random network is also presented and compared with ENL and WS networks. It turns out that topological randomness significantly simplifies the phase portrait, by reducing the number of attractors.

Some indications for further work and some critical warnings are given in section 6, where it is also observed that the ENL algorithm is particularly well suited to study the influence of "small-worldness" on the behavior of random boolean networks, where maintaining the same k value is extremely important.

2. The equal number of links network

Let us first recall for completeness some well-known definitions.

A graph $G = \{V, E\}$ is a set of nodes V and links E. The networks in this paper are graphs where every node is labeled, $v = 1, \ldots, n$. They are also simple, meaning that every edge is between distinct points, every pair of points has at most one edge. We also assume that the networks are connected. If the pair of vertices is ordered, then the graph is ordered (in this paper we only consider unordered graphs). Two vertices are adjacent if there is a link connecting them. For a recent review on regular, random, and small-world graphs, see [1].

Three global quantities play a major role in describing the properties of a graph: the average connectivity per node (k), the characteristic path length (L), and the clustering coefficient (C). They are defined as follows.

Let k_v be the connectivity of v, that is, the number of links which connect node v to other nodes in graph G. The average of k_v over all of the nodes is the average degree of connectivity k of graph G.

A path between node a and node b is defined as an ordered set of links which start from node a and end in node b (or *vice versa*) such that any node is traversed only once. The length of a given path is the number of links it contains. The distance between a pair of nodes a and b, $d(a, b)$, is the length of a shortest path between the two nodes. For each vertex v let d_v be the average of $d(v, a)$ taken over all the other $n - 1$ nodes. The characteristic path length L of graph G is then the median of d_v taken over all the vertices [3].

The neighborhood $\Gamma(v)$ of node v is the subgraph consisting of the nodes adjacent to v (excluding v itself). Let ϵ_v be the number of edges in $\Gamma(v)$, that is, the number of links connecting nodes that are both adjacent to v. The clustering coefficient γ_v of node v is the ratio between this number and the maximum number of possible edges in $\Gamma(v)$. The clustering coefficient C of graph G is then defined as the average of γ_v, taken over all the nodes of the graph.

A graph is said to be (*i*) simple, if multiple edges between the same pair of nodes are forbidden; (*ii*) sparse, if $k \ll n$; and (*iii*) connected, if any node can be reached from any other node by means of a path consisting of a set of edges (there are no separate islands).

The topology of CA is that of simple, sparse connected graphs that are spatially regular, that is, they are d-dimensional lattices. A one-dimensional ring with degree of connectivity k (see Figure 1) is characterized by [3]:

$$L = \frac{n(n + k - 2)}{2k(n - 1)} \approx \frac{n}{2k} \qquad C = \frac{3}{4}\frac{k - 2}{k - 1}. \tag{1}$$

The natural counterparts of d-dimensional lattices are random graphs, which are defined by the fact that any pair of vertices is connected by a link with a fixed probability p. In sparse random graphs ($p \ll 1$) both L and C take small values for large n (for a thorough discussion see [1]). One finds that the distribution of the number of connections per node is approximately poissonian, that is, $L \simeq \log(n)/\log(k)$ and $C \simeq p = k/n$.

We present here an algorithm that introduces some random long-range connections, but that does not change the connectivity of any node in the system. In this way we are able to introduce a topological perturbation into the system without changing any other system parameters (number of connections, transition function, etc.). In this sense, we refer to this as a "minimal perturbation algorithm." We show that

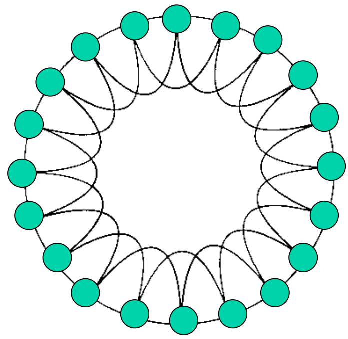

Figure 1. A regular lattice (one-dimensional ring with $k = 4$).

it displays the small-world phenomenon for a wide range of values of the fraction of redirected links.

The algorithm, for an arbitrary undirected graph, involves the following steps (see Figure 2).

1. Randomly select two nodes (called A_1 and A_2) that are not directly connected to each other (the precise selection procedure is discussed later).

2. For each node, select one of its neighbors, B_1 or B_2.

3. Check for the existence of a path connecting B_1 with A_2 that does not pass through B_2 or A_1, and another path connecting B_2 with A_1 that does not pass through B_1 or A_2. If these paths do not exist, return to step 1.

4. Add an edge between A_1 and A_2, and an edge between B_1 and B_2.

5. Delete the edges between A_1 and B_1, and between A_2 and B_2.

This algorithm is iterated until the desired number of redirections is achieved.

Let us remark that the algorithm can be used for any kind of connected graph, either regular or not, leaving the connectivity of each node unchanged. If the graph were directed, a similar algorithm with three cutting points instead of two could be applied (not discussed in this paper).

A further remark is in order: as the fraction f of rewired links grows, a random selection of the two nodes A_1 and A_2 with uniform probability tends to be ineffective. There is a high chance to pick up nodes which had been chosen previously, and to leave some connections unaltered. In order to avoid this, Watts used an algorithm in [3] which guarantees that each link is rewired when $f = 1$ (described in detail in section 3). In order to bias the choice in favor of those nodes that have a low number

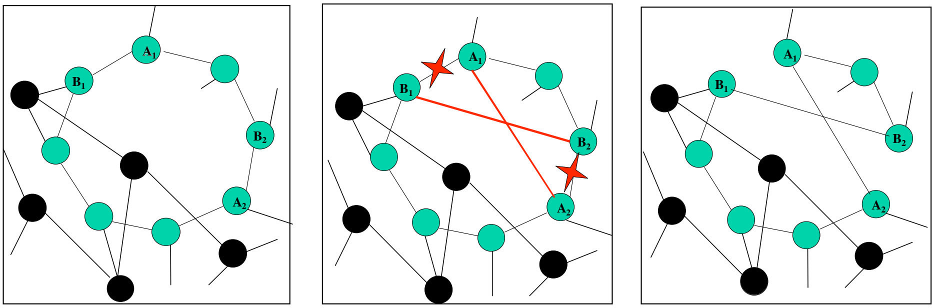

Figure 2. The algorithm of minimal perturbation with two cutting points. The nodes belonging to the selected path are highlighted.

of previous selections, our procedure comprises the following steps (the "selection number" of a node is defined as the number of times it has already been selected in the rewiring algorithm).

(a) Find the node with the smallest selection number and initialize a threshold with this number.

(b) Mark as candidate type "A" the nodes whose selection number is equal to the threshold, and count the corresponding nodes of type "B" (nodes that belong to the neighborhood of candidate type "A" nodes).

(c) If the total number of candidate nodes of type "B" is less than 2, add one unity to the threshold and return to step (b).

(d) Select at random A_1 and A_2 among the nodes marked as candidate type "A," with uniform probability.

The ENL algorithm rewires pairs of links, instead of single ones, therefore introducing correlations among the connections. Indeed, even when the fraction of rewired links is high, the nodes connected to a given node are not completely random. This can be demonstrated by considering the average distance of neighboring nodes. Let $\delta(v, w)$ be the value of the distance between vertices v and w that would have been found if it were measured on the original (regular) ring in the case $k = 2$. $\delta(v, w)$ is equal to the number of nodes between v and w plus one, and is a "natural" distance for one-dimensional rings. Then, for any network, Δ_v is defined as the average of $\delta(v, v')$ taken over the nodes v' connected to v (i.e., those in $\Gamma(v)$), and Δ is defined as the average of Δ_v over all vertices v.

If the connections were completely random, the average Δ would be close to $N/4$ for large N (the probability of any distance between two randomly chosen nodes being flat between 1 and $N/2$), while the ENL model remains definitely below this value, even when the fraction of redirected links f reaches 1.2 (see Figure 3). Note that f can be greater

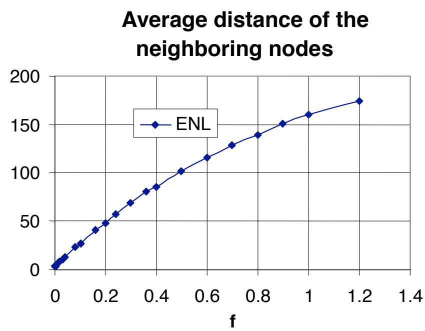

Figure 3. Average distance from neighbors for the ENL model with $n = 1000$ and $k = 10$. Each point is the average of the values obtained from 10 different networks. The value for a random graph with $n = 1000$ would be close to 250.

than one since the algorithm can be applied an arbitrary number of times.

In a sense, the existence of correlations is the price that is paid for leaving the connectivity of each node unchanged.

In order to verify the small-world properties, we performed a series of simulations based upon the perturbation of a one-lattice with a fixed connectivity degree for each node, measuring both L and C as a function of the fraction f of links which have been subject to the redirection algorithm as described. Using 1000 nodes, the tests were run for connectivities ranging between 4 and 10. Typical results are shown in Figure 4.

It is interesting to observe that already for very small f values, where the clustering remains high (and where most of the links are regular), the introduction of a few random connections leads to a drastic decrease in the characteristic path length L.

3. Comparison with the Watts–Strogatz model

The WS model [2, 3] is the first example of a small-world network generated by perturbing a regular lattice. Their procedure follows.

1. Choose each node i in turn, along with the link that connects it to its nearest neighbor in a clockwise direction $(i, i + 1)$.

2. With probability f, delete the link between nodes i and $i + 1$ and create a connection between node i and another node j randomly selected with uniform probability (excluding self- and multiple-connections).

3. When all nodes have been considered once, repeat the procedure for the links that connect each node to its next-nearest neighbor (i.e., $i + 2$), and so on.

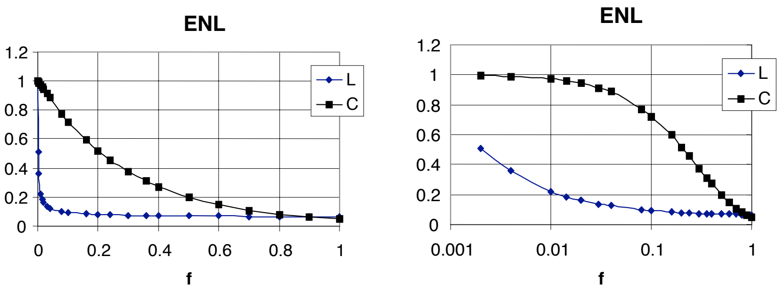

Figure 4. Normalized characteristic path length L_f/L_0 and clustering coefficient C_f/C_0 *versus* fraction of redirected links f wth $n = 1000$ and $k = 10$ (the right shows a magnification of the small-world region). Each point is the average of the values obtained from 10 different networks.

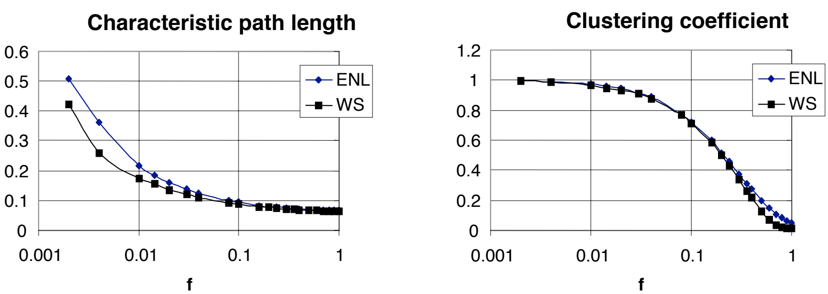

Figure 5(a). Comparison between the values of normalized characteristic path length L_f/L_0 and clustering coefficient C_f/C_0 for the ENL and WS models with $n = 1000$ and $k = 10$. Each point is the average of the values obtained from 10 different networks.

The procedure is iterated until $k/2$ rounds are completed, in order to visit all the links exactly once.

It is well known that this model gives rise to a network which, for a range of values of f, is of the small-world type, while the distribution of connectivity values $p(k)$ is poissonian [1–3].

Figure 5(a) shows a comparison between the values of C and L for the ENL and WS models. It can be seen that, in the latter case, the characteristic path length starts to decline at smaller values of f, while the clustering coefficients are very close, for $f < 0.1$, in the two models. It can therefore be seen that the small-world region is slightly larger in the WS than in the ENL case.

In order to measure the network topological properties, Watts [3] considered the number of shortcuts (i.e., links that connect nodes that do not have other common neighbors) and of contractions (i.e., the

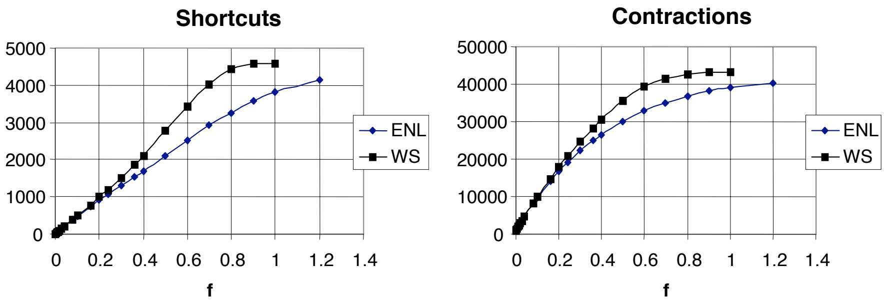

Figure 5(b). Comparison between the number of shortcuts (left) and the number of contractions (right) for the ENL and WS models with $n = 1000$ and $k = 10$. Each point is the average of the values obtained from 10 different networks.

number of pairs of unconnected vertices that have only one common neighbor) that are present in the network. In Figure 5(b) these two quantities are shown as a function of f for the two models. Note that in ENL networks, both fractions continue to grow even if $f > 1$. Recall that the ENL algorithm may rewire a link which had previously been rewired, so values of f greater than one allow a more complete randomization, while in the WS model the value $f = 1$ guarantees that all the links have been rewired.

In order to better understand the difference between the two models, it is appropriate to compare the average distance of the neighboring nodes Δ, defined in section 2. From Figure 6 one can see that the WS model reaches the Δ value for a completely random graph (i.e., $N/4$) for $f = 1$, while the ENL model remains definitely below this value for f up to 1.2. As discussed previously, this is due to the fact that the neighborhoods of the ENL model are not completely random, because rewiring pairs of links, instead of single ones, introduces correlations between different links.

4. The behavior of the majority rule

As mentioned in section 1, we now consider the influence of modifying the topology in the case of time-discrete boolean automata following the majority rule with asynchronous updating, which is defined as follows.

To each node i of the network a dynamical variable x_i is associated, which can take either the value 0 or 1. At each time step, a node is chosen at random for updating, let it be the one labeled by q (all other nodes remain unchanged). Then the value of x_q at time $t + 1$ is equal to the state of the majority of its neighbors (i.e., of the nodes belonging to $\Gamma(q)$, q itself being excluded) at time t; in case of parity, x_q is left unchanged. In formulae (recall that k is the cardinality of $\Gamma(i)$ for every

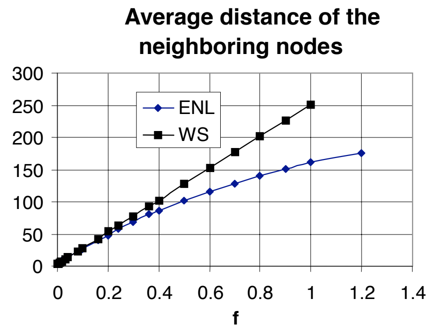

Figure 6. Comparison between average distance from neighbors for the ENL and WS models with $n = 1000$ and $k = 10$. Each point is the average of the values obtained from 10 different networks.

node):

$$x_q(t+1) = \begin{pmatrix} 1, & \text{if } \sum_{j\in\Gamma(q)} x_j(t) > \frac{k}{2} \\ 0, & \text{if } \sum_{j\in\Gamma(q)} x_j(t) < \frac{k}{2} \\ x_q(t), & \text{if } \sum_{j\in\Gamma(q)} x_j(t) = \frac{k}{2} \end{pmatrix}. \tag{2}$$

Formally, the majority rule might be considered as a particular case of a boolean neural Hopfield model [24, 25], where all the symmetric weights W_{qj} take either the value 1 (if q belongs to $\Gamma(i)$) or 0 and all the thresholds are set at $k/2$. Therefore, like in the Hopfield model, there is a Lyapunov function which guarantees that, in the case of asynchronous updating, the only stable attractors are fixed points. This feature simplifies the analysis of the modifications due to the topology, while it may limit the generalizability of the observations reported here.

Let us first discuss the case of a regular one-dimensional ring (i.e., $f = 0$). In this case the system admits several attractors, which can be described as resulting from a division of the lattice into different domains of "all 0" or "all 1" states. Recall that the dynamics follow a deterministic rule, so there is no thermal noise that tests the stability of metastable fixed points. We will call U_0 (U_1) the uniform state where all the variables take the value 0 (1).

Let r be the fraction of ones in the initial conditions; for obvious symmetry reasons, we can limit our discussion to the interval $r \in [0, 1/2]$. By performing experiments with 1000 different initial conditions (which is admittedly a very limited subset of the $2^{1000} \simeq 10^{300}$ possible initial conditions) one observes that, if r is very small, all the initial states tend to the U_0 attractor. By increasing r, one observes that other attractors are also reached and, in experiments with $r = 0.2$, it may happen that some hundreds of attractors are actually reached (Table 1). An interesting feature is that all these attractors share a large common part with U_0,

	$f = 0$		$f = 0.08$		$f = 0.8$	
$r = 0.5$	Ave.	St. Dev.	Ave.	St. Dev.	Ave.	St. Dev.
A_{1000}	1000	0	1000	0	298	14
B_{1000}	1	0	1	0	365	13
S_0	500	2	499	3	500	4
H_{med}	501	1.0	499.6	1.6	501	4.0
	$f = 0$		$f = 0.08$		$f = 0.8$	
$r = 0.2$	Ave.	St. Dev.	Ave.	St. Dev.	Ave.	St. Dev.
A_{1000}	196	176	37	58	1	0
B_{1000}	313	303	721	230	1000	0
S_0	955	24	966	10	1000	0
H_{med}	14	4	12	3	...	...

Table 1. Data about the dynamical properties of the systems at different degrees of randomness f (fraction of redirected links). Data concerning simulations performed on 10 networks of 1000 nodes with 1000 initial conditions each are shown. A_{1000} is the number of final states reached from 1000 random initial conditions. B_{1000} is the number of initial conditions (out of 1000) which belong to the largest basin of attraction. S_0 is the number of nodes (out of 1000) which always take the value 0 in the final state, in a given network. H_{med} is the average Hamming distance between two different fixed points.

that is, most of their nodes (>90%) take the same value 0. This can also be checked by considering that the Hamming distances among these attractors are small (Figure 7). So, indeed, a large number of attractors are reached in this region, but they are very similar to the uniform U_0 attractor, with some added perturbation or "disorder."

When $r = 0.5$, the initial state has no prevalence of either 0 or 1: here every initial condition out of 1000 almost always leads to a different fixed point. However, these are much different from each other, as can be seen by examining their Hamming distances, which take values close to 500 (i.e., the average distance between two random vectors of 1000 elements each, see Table 1 and Figure 7).

Let us now look at the case where a fraction of links have been rewired according to the ENL algorithm of section 2. The introduction of long-range connections indeed modifies the dynamical behavior. As can be seen in Table 1 when $r = 0.2$ but $f = 0.8$, one finds that only the U_0 attractor is reached: the presence of long-range connections eliminates the "cloud" of attractors similar to U_0 reached in the case of a regular topology.

When $r = 0.5$ the average number of attractors reached from 1000 random initial conditions shows a large decrease between $f = 0.4$ and $f = 0.8$ (Figure 8). By considering the distribution of Hamming distances, one observes that in the $f = 0.8$ case there are the uniform attrac-

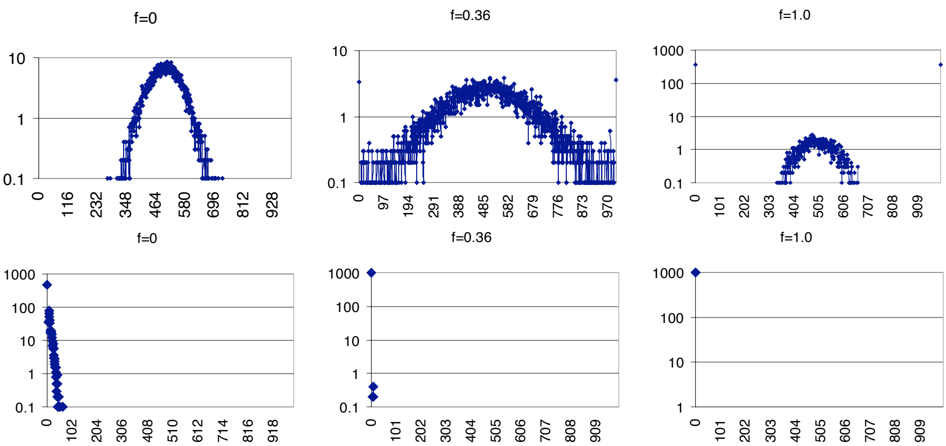

Figure 7. ENL redirection algorithm: typical behavior of attractors' Hamming distances from attractor U_0 *versus* the fraction of redirected links. The upper series refers to initial conditions with $r = 0.5$, the lower one to initial conditions with $r = 0.2$.

tors U_0 and U_1 (with fairly large basins of attraction, see Table 1), and a number of other attractors whose Hamming distances are distributed around 500 (see also Figure 7 for the $f = 1$ case). The uniform attractors are not surrounded by a cloud of slightly different fixed points, but there are still several attractors which are markedly different from U_0 and U_1.

It is also interesting to consider the case of a graph which is closer to the regular one, for example, with $f = 0.08$ (Table 1). Here one can see that the dynamical behavior is already affected by the introduction of a few random long-range connections. At $r = 0.2$ the number of fixed points is reduced, with respect to the regular case, and the basin of attraction of U_0 is larger. The cloud of similar attractors has shrunk, but has not yet disappeared as in the $f = 0.8$ case. Moreover, the average number of domains of this zone is already close to the random graph value (Figure 8).

In the case of a highly-randomized lattice ($f = 1.6$, data not shown) the number of different attractors reached from 1000 random initial conditions starts to grow at a much higher r value (beyond 0.4) than in the regular case. The peak is reached around $r = 0.5$ and corresponds to a number of attractors that is much smaller than in the regular case. In the case $r = 0.5$ the two major attractors, with equal basins of attraction, together cover about 80 percent of the initial conditions.

Finally, it is also interesting to observe the behavior of the transients as a function of the degree of randomness f. At high k values their duration increases with increasing f (as might be intuitively expected, since the introduction of many long-range connections interferes with

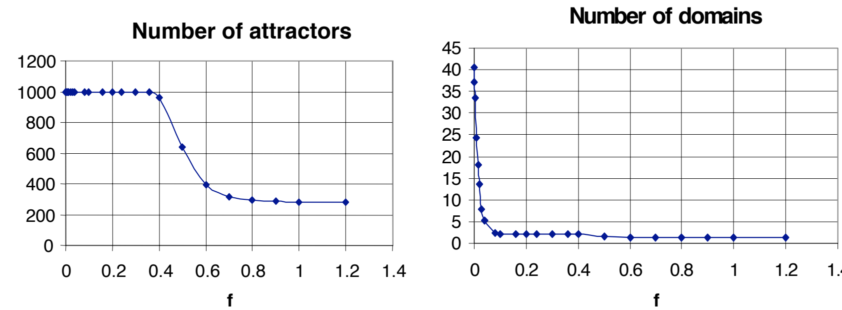

Figure 8. ENL redirection algorithm and majority rule: average number of attractors (left) and of homogeneous domains (right) *versus* the fraction f of redirected links in the case $r = 0.5$. Each point is the average of the values obtained from 10 different networks with 1000 different initial conditions for each net.

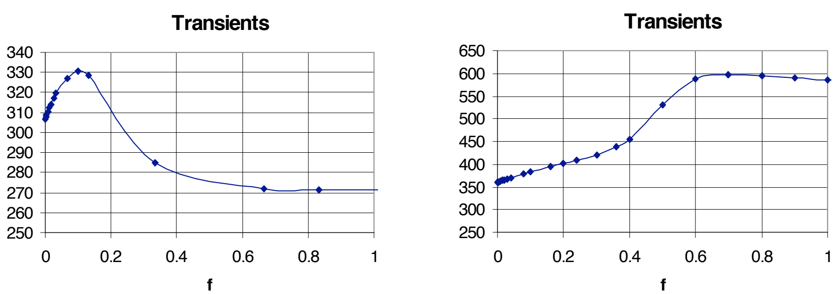

Figure 9. ENL rewiring algorithm: behavior of the transients as a function of the degree of randomness at $k = 6$ (left) and $k = 10$ (right). Each point is the average of the values obtained from 10 different networks with 1000 different initial conditions for each net.

the establishment of local domains largely independent from the configurations which take place far away). However, at smaller k values, the transient duration displays a maximum for an intermediate value of f, and then declines, as shown in Figure 9.

5. Comparing the behavior of the majority rule with different networks

The dynamical behavior of the ENL network has been extensively analyzed in section 4, while here we provide comparisons with the dynamical behavior of other models. We first consider the WS model. In this case, the average number of attractors decreases with increasing f (Figure 10) and, as can be expected, the size of the largest basin of attraction grows.

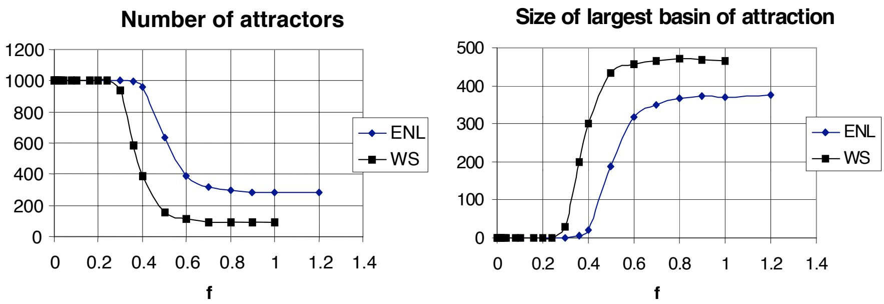

Figure 10. Comparison between number of attractors (left) and the size of the largest basin of attraction (right) for the ENL and WS models with $n = 1000$ and $k = 10$. Each point is the average of the values obtained from 10 different networks.

By comparison with the data in section 4, it can be observed that the number of attractors in WS is smaller than that in ENL at the same f level. Moreover, the number of attractors in WS at high f values is about 1/3 of the corresponding number in ENL.

These differences might be attributed to two reasons (recall that the reduction in the number of attractors is a hallmark of randomness). First, in WS the randomness of the connections is always higher, for a given value of f, as shown by the data on the average distance of neighboring nodes and discussed in section 3. Second, there is a distribution of node connectivities which introduces further randomness. While it is impossible to determine the relative weight of the two effects, it can be conjectured that both may be present and contribute to the reduction in the number of different attractors.

It is also interesting to observe that the number of different domains in WS is smaller than that which is found in ENL networks for the same f (Figure 11). This is expected due to the relationship between the number of domains and the number of different attractors. Finally, let us observe (Figure 11) that transients last longer in WS than in ENL: since updating is asynchronous, it might be expected that the average number of steps required to reach a fixed point from a random initial state is higher if there are fewer fixed points.

We also performed simulations of the majority rule on networks generated according to the Barabasi–Albert (BA) algorithm [1, 4]. The networks were generated starting with 10 initial nodes, and allowed to grow up to 1000 nodes. The average connectivity per node was 9.9, while the average values for L and C were 2.6 and 0.03, respectively. In this case there is no analogue of the fraction of redirected links, but the degree of randomness is built in during the network construction procedure.

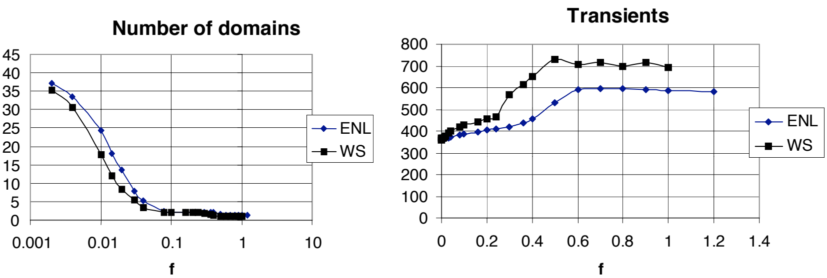

Figure 11. Comparison between number of domains (left) and the transient lengths (right) for the ENL and WS models with $n = 1000$ and $k = 10$. Each point is the average of the values obtained from 10 different networks.

The behavior of the BA network is markedly different from those of the ENL and WS models. The uniform atttractors dominate the phase portrait, and for any initial condition (even if $r = 0.5$) the system almost always reaches either U_0 or U_1. The average number of attractors per network (with 1000 random initial conditions, as in the previous cases) is actually 3.1, but the basins of the other attractors that can be found are very small.

Therefore one might conclude that the topological randomness of the BA model is higher than those of both WS and ENL, and that topological disorder plays a role analogous to that of random noise, by eliminating a great majority of the very many attractors which exist in the regular lattice.

In order to provide a further test of this latter hypothesis, we performed simulations of the majority rule using a fully random (FR) network with fixed connectivity k. In a FR network each node is connected to k other nodes chosen at random with uniform probability, avoiding double connections. In this case one indeed finds only the two uniform attractors, therefore confirming the role of topological disorder in this kind of model. A detailed comparison of the properties of the four models which have been tested is summarized in Table 2.

6. Conclusions

Further work is needed to explore the effects of topological perturbations in the case where different transition functions are used, and in order to outline a theory of the dynamical effects of these perturbations. It has been observed that topological modifications may profoundly affect the dynamics of discrete systems like random boolean networks [9–11] and of continuous systems like interacting oscillators [8].

property	ENL (f = 1)	WS (f = 1)	BA	FR
L	3.3	3.3	2.6	3.2
C	0.03	0.01	0.03	0.01
N_S	3800	4600	3800	4600
N_C	39000	43000	140000	41000
A_{1000}	280	90	3	2
N_D	1.3	1.1	1+	1
T_t	600	700	650	830

Table 2. Comparison of topological and dynamical properties of different models (the ENL and WS models refer to the case $f = 1$). All data refer to tests with 10 different networks with 1000 nodes and connectivity $k = 10$ ($k = 9.9$ in the BA case). Dynamical analysis has been performed by testing 1000 different initial conditions for each network. L is the characteristic path length, C the clustering coefficient, N_S the average number of shortcuts, N_C the average number of contractions, A_{1000} the number of attractors found, N_D the average number of domains, and T_t is the average duration of transients. The average number of domains in the BA model is slightly larger than one, due to the presence of a few attractors (one per network on average). The actual value found in the tests is 1.001, which we do not show in the table because it would imply that the fourth digit were significant. However, we use "1+" to distinguish this case from that of the fully random network, where N_D was exactly one in all the tests.

The results reported here demonstrate that the introduction of long-range connections (without changing the number of connections of every node) can have a profound effect on the dynamics of the majority rule. In the case of a small-world network, it has also been shown that major dynamical features are affected at a fairly small fraction of redirected links.

In the particular case considered here, it was observed that these changes lead, with respect to the regular case, to a decrease in the number of attractors that are reached, which can be observed even at small f values. Qualitatively, one sees a "cloud" of attractors corresponding to minor perturbations of the major, uniform attractors disappear. Similar results could be obtained by modifying the evolution rule with the addition of a finite noise level, which tests the stability of the many metastable attractors. One may guess that the similarity between the effects of the introduction of topological disorder into a deterministic model and the introduction of stochastic dynamics might hold also for a broader class of models, although clarifying this point requires further work. It should also be stressed that topological disorder is not required to observe stochasticity in discrete deterministic systems [26].

In the case discussed in this paper, an effect on the number and features of the attractors somehow similar to that of increasing f might be obtained by taking larger neighborhoods, that is, larger k values.

This is also an observation whose generality could be tested on a wider set of cases.

A price has to be paid for keeping the number of connections fixed for every node. This can be shown by comparing the average distance between neighboring nodes in the Watts–Strogatz (WS) and equal number of links (ENL) models. At the same value of the fraction f of redirected links, there is "less randomness" in the latter one, because of the correlations that are created by the pairwise process of link modification.

On the other hand, it is worth stressing again the importance of separating the effects of introducing long-range connections from those of introducing nodes with different connectivities.

While the ENL algorithm described here works for undirected networks, generalization to directed graphs is possible (using a three-point instead of a two-point method) and will be described elsewhere.

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