Power Spectral Analysis of Elementary Cellular Automata

Shigeru Ninagawa*
Division of Information and Computer Science,
Kanazawa Institute of Technology,
7-1 Ohgigaoka, Nonoichi, Ishikawa 921-8501, Japan

Spectral analysis of elementary cellular automata is performed. A power spectrum is calculated from the evolution of 88 independent rules starting from random initial configurations. As a result, it is found that rule 110 exhibits $1/f$ noise during the longest time steps. Rule 110 has proved to be capable of supporting universal computation. These results suggest that there is a relationship between computational universality and $1/f$ noise in cellular automata.

1. Introduction

Cellular automata (CAs) are spatially and temporally discrete dynamical systems with large degrees of freedom. Since spectral analysis is one useful method for investigating the behavior of dynamical systems [1], it is reasonable to apply it to studying the behavior of CAs. In this paper we deal with elementary CAs (ECAs), namely one-dimensional and two-state, three-neighbor CAs. ECAs have been investigated in detail for their simplicity ([2], appendix in [3]). Although the ECA rule space is not large, there is a wide variety of rules which exhibit regular, chaotic, or complex behavior. Spectral analysis was used for the investigation of the spatial structures produced by ECAs in connection with regular languages in [4], whereas we focus on the temporal behavior of ECAs.

2. Spectral analysis of elementary cellular automata

Let $x_i(t)$ be the value of site i at time step t in an ECA. The value of each site is specified as 0 or 1. The site value evolves by iterating the mapping

$$x_i(t+1) = F(x_{i-1}(t), x_i(t), x_{i+1}(t)). \tag{1}$$

Here F is an arbitrary function specifying the ECA rule. The ECA rule is determined by a binary sequence with length $2^3 = 8$,

$$F(1,1,1),\ F(1,1,0), \ldots,\ F(0,0,0). \tag{2}$$

*Electronic mail address: ninagawa@infor.kanazawa-it.ac.jp.

Therefore the total number of possible distinct ECA rules is $2^8 = 256$ and each rule is abbreviated by the decimal representation of the binary sequence (2) as used in [2]. A parameter λ is defined as the density of ones in the binary sequence (2) [5]. Out of the 256 ECA rules, 88 of them remain independent (appendix in [6]). With a few exceptions, we use as representative rules those with smaller λ values and the smaller decimal rule numbers adopted in [6].

The discrete Fourier transform of the time series of state $x_i(t)$ for $t = 0, 1, \ldots, T-1$ is given by

$$\hat{x}_i(f) = \frac{1}{T}\sum_{t=0}^{T-1} x_i(t)\exp\left(-i\frac{2\pi tf}{T}\right). \tag{3}$$

It is natural to define the power spectrum of a CA as

$$S(f) = \frac{1}{N}\sum_{i=1}^{N} |\hat{x}_i(f)|^2, \tag{4}$$

where N means array size and the summation is taken over all cells in the array. The power $S(f)$ at frequency f intuitively means the "strength" of the periodic vibration with period T/f in the evolution of observation length T. Another definition of power spectrum is given by calculating spatial density before the Fourier transform. It might be useful to investigate the behavior of CAs with an oscillating density in time, although we do not focus on them in this research.

Since we are interested in the dynamics that a CA rule brings, rather than the behavior from a particular initial configuration which is elaborately designed, throughout this paper we use random initial configurations in which each site takes state 0 or 1 randomly with independent equal probabilities. Generally, the evolution of one-dimensional CAs tends to depend more heavily on an initial configuration than that of two-dimensional CAs. So we use some different initial configurations to grasp the typical behavior of a CA. In addition, an array size larger than 400 seems required to avoid singular behavior. We adopt periodic boundary conditions where each end of the array is connected like a ring.

Spectral analysis revealed that 88 independent ECAs can be classified into several categories according to the shape of their power spectra. The results are as follows.

2.1 Category 1: Extremely low power density

The power spectra in this category are characterized by an extremely low power density at almost all frequencies. Figure 1 shows the space-time patterns (left) and the power spectra (right) of rule 32 (top) and rule

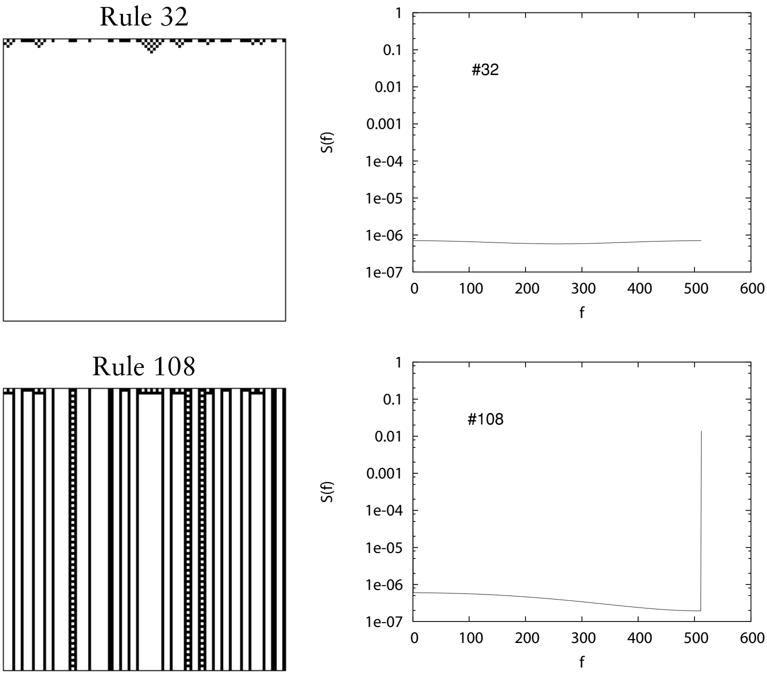

Figure 1. Space-time patterns (left) and power spectra (right) of rule 32 (top) and rule 108 (bottom). The space-time patterns consist of 100 cells for 100 time steps. The power spectra are calculated from the evolution of 700 cells for 1024 time steps.

108 (bottom). Space-time patterns show configurations obtained at successive time steps on successive horizontal lines in which black squares represent sites with value 1, white squares value 0. The power spectrum is calculated from the evolution of 700 cells for 1024 time steps. Only half of the components of the spectrum are shown since the other half are redundant. The y-axis is plotted on a logarithmic scale. The power density in the power spectrum of rule 32 is extremely low at all frequencies except $f = 0$. This is ascribed to the initial transient behavior which vanishes after the first few time steps. Other rules with similar power spectra are rules 0, 4, 8, 12, 13, 26, 40, 44, 72, 76, 77, 78, 104, 128, 132, 136, 140, 160, 164, 168, 172, 200, and 232. These rules are considered to be in class I, or a part of class II, with evolutions that lead to stable structures according to Wolfram's classification scheme in [7].

The power spectrum of rule 108 has a peak at $f = 512$ which is caused by periodic structures with period 2. Rules like this are a part of class II with evolutions that lead to periodic structures, namely rules 1, 5, 19, 23, 28, 29, 33, 37, 50, 51, 133, 156, and 178.

2.2 Category 2: Broad-band noise

The power spectra in this category are characterized by broad-band noise. In general, broad-band noise in a power spectrum means that the original time series is nonperiodic. This category is divided into two subcategories. A typical example of the first subcategory 2-A is shown in the top of Figure 2. The power spectrum of rule 6 has a higher power density at all frequencies compared with those in category 1. The space-time pattern of rule 6 yields uniformly shifting structures which move one cell left every time step. So the evolution with array size N becomes periodic with period N under periodic boundary conditions. Therefore the power spectrum with observation length $T \gg N$ has a fundamental peak component at $f = T/N$ and several harmonic peak components, while the power spectrum with $T \leq N$ is similar to white noise, which means the time series is random. But its randomness is mainly included in an initial configuration rather than generated by a rule. Subcategory 2-A consists of rules 2, 3, 6, 7, 9, 10, 11, 14, 15, 24, 25, 26, 27, 34, 35, 38, 41, 42, 43, 46, 56, 57, 58, 74, 130, 134, 138, 142, 152, 154, 162, 170, and 184.

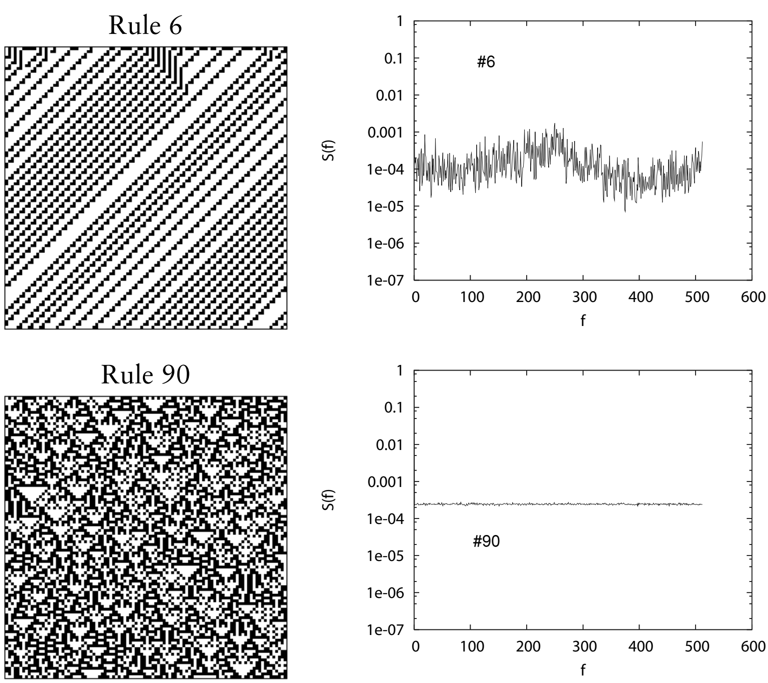

Figure 2. Space-time patterns (left) and power spectra (right) of rule 6 (top) and rule 90 (bottom).

The power spectra in subcategory 2-B are characterized by white noise. The bottom of Figure 2 shows the space-time pattern (left) and the power spectrum (right) of rule 90. The power spectrum in subcategory 2-B has an almost equal power density at all frequencies except for rules 18 and 146, which have a peak with period 2. This result implies that evolution in subcategory 2-B is virtually orderless, although it is causally determined by a rule. Randomness in the evolution of subcategory 2-B owes much to the rule itself rather than initial configurations. Subcategory 2-B consists of rules 18, 22, 30, 45, 60, 90, 105, 106, 129, 146, 150, and 161 which coincide with class III [6].

By using an initial configuration containing a single site with value 1, the difference between subcategories 2-A and 2-B becomes clear. Subcategory 2-A resembles category 1 in the shape of its power spectrum with the exceptions of rules 26 and 154, which exhibit broad-band noise. The power spectrum in subcategory 2-B, however, remains broad-band noise with the exception of rule 106, having an extremely low power density. Rules 26, 15, and 106 are considered as intermediates between these two subcategories.

2.3 Category 3: Power law spectrum

The power spectrum in this category is characterized by power law at low frequencies. Figure 3 shows the space-time patterns (left) and the power spectra (right) of rules 54 (top), 62 (middle), and 110 (bottom). The x and y axes in the power spectra are plotted on a logarithmic scale and the broken line represents the least square fitting of the power spectrum from $f = 1$ to $f = 10$ by $\ln S(f) = \alpha + \beta \ln f$, with $\beta = -0.5$ for rule 54, $\beta = -1.2$ for rule 62, and $\beta = -1.5$ for rule 110. Rules 54 and 110 are considered to be in class IV, which is supposed to be capable of supporting universal computation [7] while there is a guess that the evolution of rule 62 is too simple to be universal [8].

2.4 Exceptional rules

There are two exceptional rules which do not belong to any of the three categories. Rule 73 is called "locally chaotic" because the array is divided into some independent domains by stable "walls" and chaotic patterns are generated in each domain [6]. Each domain has an individual period which contributes the peak in the power spectrum. Figure 4 shows the space-time pattern (left) and the power spectrum (right) of rule 73. The power spectrum of rule 204 has zero power density at all frequencies except for $f = 0$ because the evolution of rule 204 retains the initial configuration.

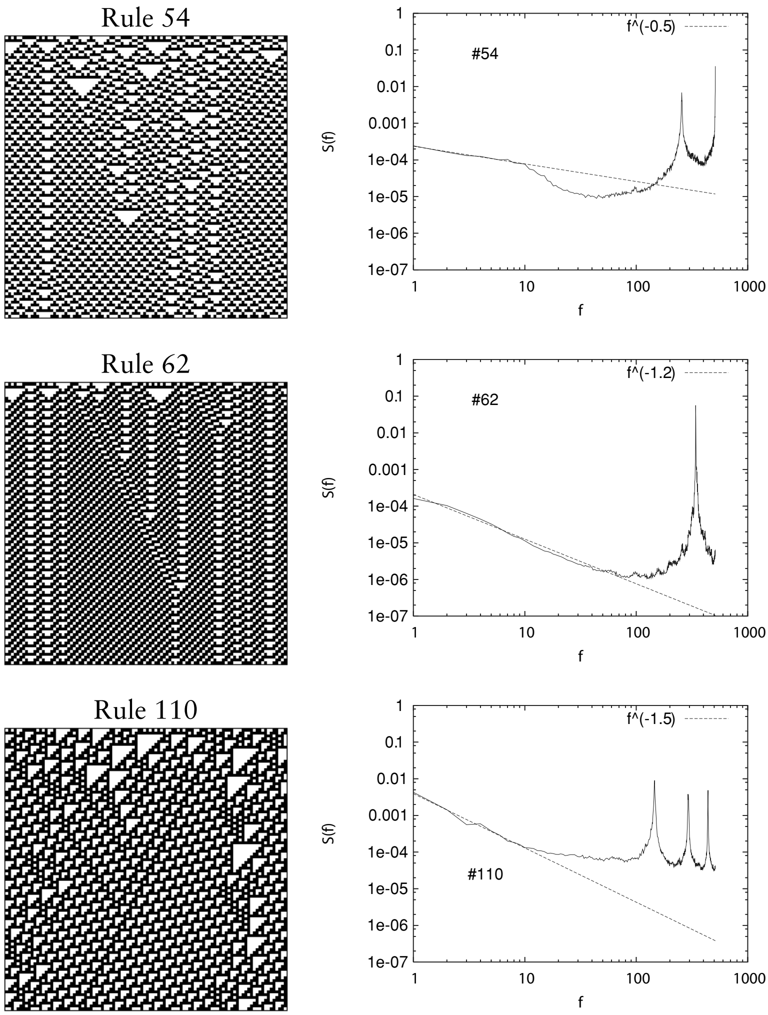

Figure 3. Space-time patterns (left) and the power spectra (right) of rules 54 (top), 62 (middle), and 110 (bottom). The x and y axes in the power spectra are plotted on a logarithmic scale. The broken line in the power spectra represents the least square fitting of the power spectrum from $f = 1$ to $f = 10$ by $\ln S(f) = \alpha + \beta \ln f$, $\beta = -0.5$ for rule 54, $\beta = -1.2$ for rule 62, and $\beta = -1.5$ for rule 110.

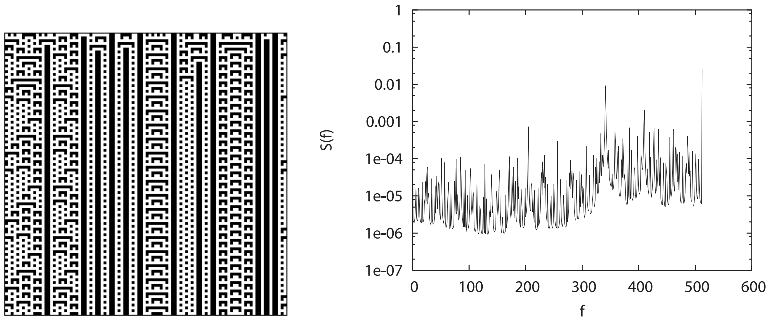

Figure 4. Space-time pattern (left) and power spectrum (right) of rule 73.

3. 1/*f* noise in elementary cellular automata

A fluctuation with a power spectrum $S(f)$ that is inversely proportional to frequency f is called $1/f$ noise [9]. $1/f$ noise can be observed in a wide variety of phenomena such as the voltage of vacuum tubes, the rate of traffic flow, and the loudness of music. However, its origin is not well understood.

The most controversial problem in $1/f$ noise is whether $1/f$ noise lasts forever or not. Generally speaking, if there is finite correlation time τ in a fluctuation, the power spectrum with an observation length of $T > \tau$ has an almost equal power density at frequencies smaller than $1/\tau$. Likewise if there is finite correlation time τ in the evolution of a CA, the power spectrum with an observation length of $T > \tau$ has an almost equal power density at frequencies smaller than T/τ. So the power spectrum becomes close to a flat line at low frequencies as the observation length T becomes longer than the correlation time τ. Moreover we can guess that the correlation time τ depends on array size N because the average transient time steps T_{ave} of rule 110, which is supposed to be relevant to τ, increases algebraically with array size N, $T_{\mathrm{ave}} \propto N^{\alpha}$, with $\alpha = 1.08$ [10].

Therefore we investigate the value of the exponent β of the power spectra $S(f) \propto f^{\beta}$ for various observation lengths and array sizes to compare the behaviors of rules 110, 54, and 62. Since one-dimensional CAs in general have larger variations in the value of β with initial configurations than two-dimensional CAs, we calculate the value of β averaged over some different random initial configurations.

Figure 5 shows the average $\langle\beta\rangle$ of the exponent estimated by the least square fitting of the power spectrum from $f = 1$ to $f = 10$ by $\ln S(f) = \alpha + \beta \ln(f)$ over 400 random initial configurations as a function of array size N and observation length T for rules 110 and 54 (top), and rules 110 and 62 (bottom). The solid line represents $\langle\beta\rangle$ of rule 110 and the broken line rules 54 (top) and 62 (bottom). $\langle\beta\rangle$ of rules 110, 54,

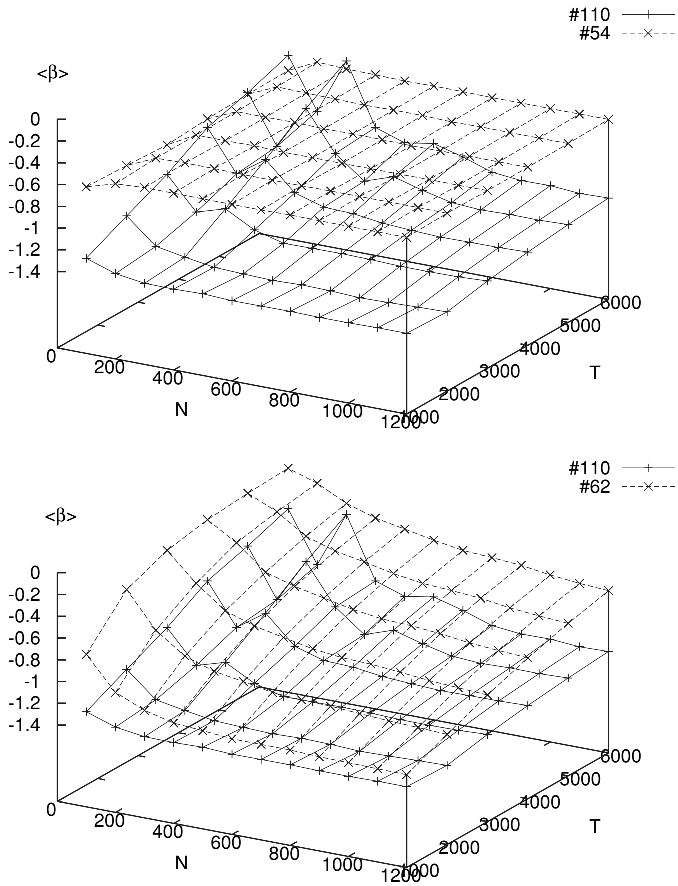

Figure 5. Average $\langle\beta\rangle$ of the exponent estimated by the least square fitting of the power spectrum from $f = 1$ to $f = 10$ by $\ln S(f) = \alpha + \beta \ln(f)$ for 400 random initial configurations as a function of array size N and observation length T for rules 110 and 54 (top), and for rules 110 and 62 (bottom). The solid line represents $\langle\beta\rangle$ of rule 110 and the broken line rules 54 (top) and 62 (bottom). $\langle\beta\rangle$ of rules 110, 54, and 62 at $N = 1200, T = 8000$ equal -1.2, -0.4, and -0.6 respectively.

and 62 at $N = 1200, T = 6000$ equal -1.2, -0.4, and -0.6 respectively. On the whole, $\langle\beta\rangle$ of rules 110, 54, and 62 increases as the array size N decreases or the observation length T increases. The reason $\langle\beta\rangle$ of rule 110 at $N = 300$ is close to zero is that singular behavior tends to occur frequently in one-dimensional CAs with small array sizes. When the array size is larger than 300, $\langle\beta\rangle$ of rule 110 is smaller than those of rules 54 and 62. This result means that the evolution of rule 110 exhibits $1/f$ noise during the longest observation length of the ECAs.

4. Origin of $1/f$ noise in rule 110

In this section we compare two examples of the evolution of rule 110 from distinct random initial configurations to study the origin of $1/f$ noise in the evolution of rule 110. Figure 6 shows a set of space-time patterns of the evolution of rule 110 from a random initial configuration of 200 cells for 1000 time steps. The first 200 time steps are shown on the left and the last 200 time steps are on the right. The evolution at first exhibits transient behavior but becomes periodic at $t = 306$ with period 750. The space-time pattern on the right shows that several gliders are moving monotonously within a periodic background. Figure 7 shows another set of space-time patterns from different random initial configurations. The evolution keeps transient behavior at $t = 999$ although it eventually becomes periodic at $t = 2416$ with period 400. The space-time pattern on the right shows that several gliders interact complexly within a periodic background. Figure 8 shows the power spectra calculated from the space-time patterns shown in Figures 6 and 7. The left one is calculated from the evolution shown in Figure 6 and the right one from Figure 7. The broken line represents the least square fitting of the power spectrum from $f = 1$ to $f = 10$ by $\ln S(f) = \alpha + \beta \ln f$, with $\beta = -0.5$ (left) and $\beta = -1.5$ (right). The left one is too flat to be considered as type $1/f$. Comparing these two examples of the evolution suggests that the difference in the exponent of power spectra at low frequencies depends on the duration of transient behavior.

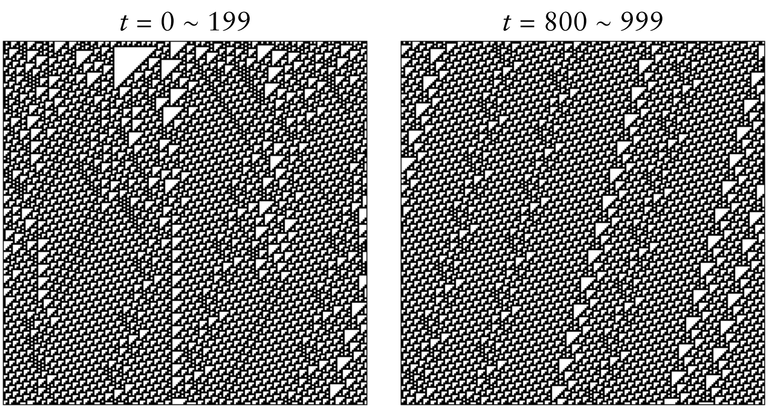

Figure 6. Space-time patterns of rule 110 from a random initial configuration. Each picture is 200 cells wide. The first 200 steps of evolution are shown on the left and the last 200 steps on the right. The evolution becomes periodic at $t = 306$ with period 750. Several gliders are moving monotonously in the last 200 time step evolution.

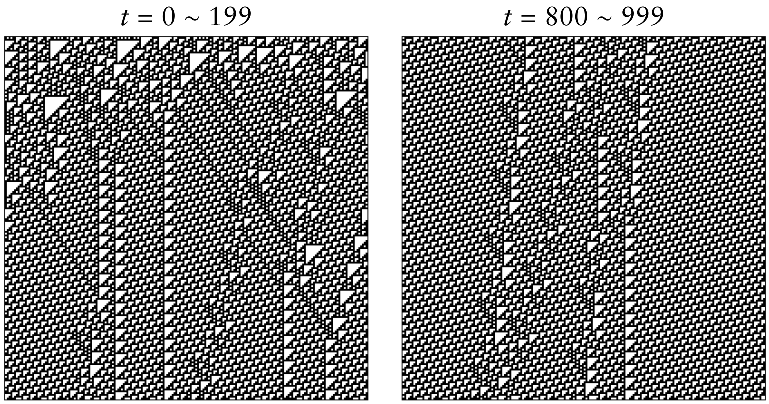

Figure 7. Space-time patterns of rule 110 from a different random initial configuration than the one used in Figure 6. Several gliders interact complexly in the last 200 steps of evolution. The evolution eventually becomes periodic at $t = 2416$ with period 400.

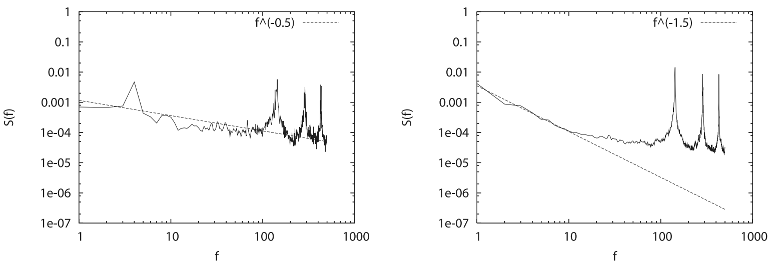

Figure 8. Power spectra of rule 110 from two random initial configurations. The left one is calculated from the evolution in Figure 6 and the right one from Figure 7. The broken line represents the least square fitting of the spectrum from $f = 1$ to $f = 10$ by $\ln S(f) = \alpha + \beta \ln f$, with $\beta = -0.5$ (left) and $\beta = -1.5$ (right).

Intermittency in chaotic dynamical systems is one of the main mechanisms of $1/f$ noise [11]. A system with a particular parameter value exhibits periodic behavior which is disrupted occasionally and irregularly by a "burst." This burst persists for a finite duration, it stops and a new periodic behavior starts. Intermittent chaos occurs when the transition from periodic to chaotic behavior takes place as the parameter is varied. During the transient behavior of rule 110 there are alternating periodic phases in which gliders are shifting in a periodic background without collisions and a burst that is caused by colliding gliders. It seems likely that the recurrence of the periodic phase and the burst in the transient behavior generates intermittency and causes $1/f$ noise in rule 110.

5. Conclusion

In this research we performed spectral analysis on the evolution of 88 independent elementary cellular automata (ECAs) starting from random initial configurations and concluded that rule 110 exhibits $1/f$ noise during the longest time steps. Rule 110 has proved to be capable of supporting universal computation [8, 12]. These results suggest that there is a relationship between computational universality and $1/f$ noise in CAs.

But there remains the problem of how to decide array size and observation length to detect $1/f$ noise in CAs. The average $\langle\beta\rangle$ of the exponent of the power spectrum of rules 110, 54, and 62 in Figure 5 at $N = 100, T = 6000$ equal -0.4, -0.5, and -0.04 respectively. These power spectra are not regarded as type $1/f$. These examples make it clear that $1/f$ noise in ECAs is observed only in a specific range of array size N and observation length T. Since the guess is that $1/f$ noise in CAs is caused by transient behavior, as mentioned in section 4, we have to estimate experimentally a proper range of array size and observation length in which the evolution remains transient to observe a type $1/f$ power spectrum.

Another example which possesses both the exhibition of $1/f$ noise and the capability of supporting universal computation is the Game of Life (Life) which is a two-dimensional and two-state, nine-neighbor outer totalistic CA [13]. It is supposed that a universal computer can be constructed on the array of Life by considering a glider as a pulse in a digital circuit. In addition, the evolution from random initial configurations has a type $1/f$ spectrum [14]. Another important property of Life is self-organized criticality where the distribution $D(T)$ of time steps required for the lattice to return to stability following a random single-site perturbation is characterized by $D(T) \propto T^{-1.6}$ [15]. Since $1/f$ noise and self-organized criticality seem to be interrelated phenomena, rule 110 might have the property of self-organized criticality as well.

It must be noted that the power law in the power spectrum of rule 110 holds for one decade in the range of frequencies ($f = 1{\sim}10$) as shown in Figure 8 (right), whereas it holds for three decades ($f = 1{\sim}1000$) in Life [14]. The periodic background which is observed in rule 110, but not in Life, accounts for this difference. The three peaks of the power spectrum of rule 110 at $f = 143, 286$, and 429 in Figure 8 (right) are the fundamental, the second, and the third harmonic components caused by the periodic background with period seven respectively. The interaction between the periodic background and other structures broadens the peaks and contributes to the power in high frequencies.

The hypothesis of "the edge of chaos" has evoked considerable controversy [16]. This hypothesis says the ability to perform universal computation in a system arises near a transition from regular to chaotic behavior such as class IV. So far various statistical quantities, such as entropy and difference pattern spreading rate, have been proposed to detect class IV quantitatively [17]. The exponent of the power spectrum at low frequencies might be another useful index.

Acknowledgments

I would like to thank the referee for instructive comments on the original manuscript.

References

[1] J. Crutchfield, D. Farmer, N. Packard, R. Shaw, G. Jones and R. J. Donnelly, "Power Spectral Analysis of a Dynamical System," *Physics Letters*, **76A** (1980) 1–4.

[2] S. Wolfram, "Statistical Mechanics of Cellular Automata," *Reviews of Modern Physics*, **55** (1983) 601–644.

[3] S. Wolfram (editor), *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).

[4] W. Li, "Power Spectra of Regular Languages and Cellular Automata," *Complex Systems*, **1** (1987) 107–130.

[5] C. G. Langton, "Studying Artificial Life with Cellular Automata," *Physica D*, **22** (1986) 120–149.

[6] W. Li and N. Packard, "The Structure of the Elementary Cellular Automata Rule Space," *Complex Systems*, **4** (1990) 281–297.

[7] S. Wolfram, "Universality and Complexity in Cellular Automata," *Physica D*, **10** (1984) 1–35.

[8] S. Wolfram, *A New Kind of Science* (Wolfram Media, Champaign, IL, 2002).

[9] M. S. Keshner, "1/*f* Noise," *Proceedings of the IEEE*, **70** (1982) 212–218.

[10] W. Li and M. G. Nordahl, "Transient Behavior of Cellular Automaton Rule 110," *Physics Letters A*, **166** (1992) 335–339.

[11] Y. Pomeau and P. Manneville, "Intermittent Transition to Turbulence in Dissipative Dynamical Systems," *Communications in Mathematical Physics*, **74** (1980) 189–197.

[12] M. Cook, "Universality in Elementary Cellular Automata," *Complex Systems*, **15** (2004) 1–40.

[13] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays, Volume 4* (A. K. Peters, Wellesley, second edition, 2004).

[14] S. Ninagawa, M. Yoneda, and S. Hirose, "1/*f* Fluctuation in the 'Game of Life'," *Physica D*, **118** (1998) 49–52.

[15] P. Bak, K. Chen, and M. Creutz, "Self-organized Criticality in the 'Game of Life'," *Nature*, **342** (1989) 780–782.

[16] C. Langton, "Computation at the Edge of Chaos: Phase Transitions and Emergent Computation," *Physica D*, **42** (1990) 12–37.

[17] W. Li, N. H. Packard, and C. G. Langton, "Transition Phenomena in Cellular Automata Rule Space," *Physica D*, **45** (1990) 77–94.